Backward stochastic differential equations and optimal control of marked point processes

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Abstract

We study a class of backward stochastic differential equations (BSDEs) driven by a random measure or, equivalently, by a marked point process. Under appropriate assumptions we prove well-posedness and continuous dependence of the solution on the data. We next address optimal control problems for point processes of general non-markovian type and show that BSDEs can be used to prove existence of an optimal control and to represent the value function. Finally we introduce a Hamilton-Jacobi-Bellman equation, also stochastic and of backward type, for this class of control problems: when the state space is finite or countable we show that it admits a unique solution which identifies the (random) value function and can be represented by means of the BSDEs introduced above.

Keywords: Backward stochastic differential equations, optimal control problems, marked point processes.

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1 Introduction

The purpose of this paper is to study a class of backward stochastic differential equations (BSDEs for short) and apply these results to solve optimal control problems for marked point processes. Under appropriate assumptions, an associated Hamilton-Jacobi-Bellman equation of stochastic type is also introduced and solved in this non-markovian framework.

General nonlinear BSDEs driven by the Wiener process were first solved in [20]. Since then, many generalizations have been considered where the Wiener process was replaced by more general processes. Among the earliest results we mention in particular [13], [14], to which some of our results are inspired, and we refer e.g. to [7] for a recent result and for indications on the existing bibliography.

We address a class of BSDEs driven by a random measure, naturally associated to a marked point process. There exists a large literature on this class of processes, and in particular to the corresponding optimal control problems: we only mention the classical treatise [6] and the recent book [5] as general references. In spite of that, there are relatively few results on their connections with BSDEs. In the general formulation of a BSDE driven by a random measure, one of the unknonwn processes (the one associated with the martingale part, or Z-process) is in fact a random field. This kind of equations has been introduced in [24], and has been later considered in [2], [23] in the markovian case, where the associated (nonlocal) partial differential equation and related non-linear expectations have been studied.

In these papers the BSDE contains a diffusive part and a jump part, but the latter is only considered in the case of a Poisson random measure. In order to give a probabilistic

representation of solutions to quasi-variational inequalities in the theory of stochastic impulse control, in [18] a more difficult problem involving also constraints on the jump part is formulated and solved, but still in the Poisson case and in a markovian framework.

To our knowledge, the only general result beyond the Poisson case is the paper [26]. Here, under conditions of Lipschitz type on the coefficients and assuming the validity of appropriate martingale representation theorems, a general BSDE driven by a diffusive and a jump part is considered and well-posedness results and a comparison theorem are proven. However, it seems that in this paper the formulation of the BSDE was not chosen in view of applications to optimal control problems. Indeed, in contrast to [24] or [2], the generator of the BSDE depends on the Z-process in a specific way (namely as an integral of a Nemytskii operator) that is generally not valid for the hamiltonian function of optimal control problems (compare for instance formula (1.3) below) and therefore prevents direct applications to these problems.

In our paper we consider a BSDE driven by a random measure, without diffusion part, on a finite time interval, of the following form:

$$Y_t + \int_t^T \int_K Z_s(y)\, q(ds\, dy) = \xi + \int_t^T f_s(Y_s, Z_s(\cdot))\, dA_s, \qquad t \in [0, T], \tag{1.1}$$

where the generator f and the final condition ξ are given.

Here the basic probabilistic datum is a marked point process (T_n, ξ_n) where (T_n) is an increasing sequence of random times and (ξ_n) a sequence of random variables in the state (or mark) space K. The corresponding random counting measure is $p(dt\, dy) = \sum_n \delta_{(T_n, \xi_n)}$, where δ denotes the Dirac measure. We denote (A_t) the compensator of the counting process $(p([0,t] \times K))$ and by $\phi_t(dy)\, dA_t$ the (random) compensator of p. Finally, the compensated measure $q(dt\, dy) = p(dt\, dy) - \phi_t(dy)\, dA_t$ occurs in equation (1.1). The unknown process is a pair $(Y_t, Z_t(\cdot))$, where Y is a real progressive process and $\{Z_t(y),\, t \in [0, T],\, y \in K\}$ is a predictable random field.

The random measure p is fairly general, the only restriction being non explosion (i.e. $T_n \to \infty$) and the requirement that (A_t) has continuous trajectories. We allow the space K to be of general type, for instance a Lusin space. Therefore our results can also be directly applied to marked point processes with discrete state space. We mention at this point that the specific case of finite or countable Markov chains has been studied in [8], [9], see also [10] for generalizations.

The basic hypothesis on the generator f is a Lipschitz condition requiring that for some constants $L \geq 0$, $L' \geq 0$,

$$|f_t(\omega, r, z(\cdot)) - f_t(\omega, r', z'(\cdot))| \leq L'|r - r'| + L \left(\int_K |z(y) - z'(y)|^2 \phi_t(\omega, dy) \right)^{1/2}$$

for all (ω, t), for $r, r' \in \mathbb{R}$, and z, z' in appropriate function spaces (depending on (ω, t)): see below for precise statements. We note that the generator of the BSDE can depend on the unknown Z-process in a general functional way: this is required in the applications to optimal control problems that follow, and it is shown that our assumptions can be effectively verified in a number of cases. In order to solve the equation, beside measurability assumptions, we require the summability condition

$$\mathbb{E} \int_0^T e^{\beta A_t} |f_t(0,0)|^2 dA_t + \mathbb{E}\, [e^{\beta A_T} |\xi|^2] < \infty,$$

to hold for some $\beta > L^2 + 2L'$. Note that in the Poisson case mentioned above we have a deterministic compensator $\phi_t(dy)\, dA_t = \pi(dy)\, dt$ for some fixed measure π on K and the summability

condition reduces to a simpler form, not involving exponentials of stochastic processes. We prove existence, uniqueness, a priori estimates and continuous dependence upon the data for the solution to the BSDE.

The results described so far are presented in section 3, after an introductory section devoted to notation and preliminaries.

In section 4 we formulate a class of optimal control problems for marked point processes, following a classical approach exposed for instance in [6]. For every fixed $(t, x) \in [0, T] \times K$, the cost to be minimized and the corresponding value function are

$$J_t(x, u(\cdot)) = \mathbb{E}_u^{\mathcal{F}_t} \left[\int_t^T l_s(X_s^{t,x}, u_s)\, dA_s + g(X_T^{t,x}) \right], \qquad v(t, x) = \operatorname*{ess\,inf}_{u(\cdot) \in \mathcal{A}} J_t(x, u(\cdot)),$$

where $\mathbb{E}_u^{\mathcal{F}_t}$ denotes the conditional expectation with respect to a new probability $\mathbb{P}_u$, depending on a control process (u_t) and defined by means of an absolutely continuous change of measure: the choice of the control process modifies the compensator of the random measure under $\mathbb{P}_u$ making it equal to $r_t(y, u_t)\phi_t(dy)\, dA_t$ for some given function r. To this control problem we associate the BSDE

$$Y_s^{t,x} + \int_s^T \int_K Z_r^{t,x}(y)\, q(dr\, dy) = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Z_r^{t,x}(\cdot))\, dA_r, \qquad s \in [t, T]. \tag{1.2}$$

where $(X_r^{t,x})$ is a family of marked point processes, each starting from x at time t, and the generator contains the hamiltonian function

$$f(\omega, t, x, z(\cdot)) = \inf_{u \in U} \left\{ l_t(\omega, x, u) + \int_K z(y)\, (r_t(\omega, y, u) - 1)\, \phi_t(\omega, dy) \right\}. \tag{1.3}$$

Assuming that the infimum is in fact a minimum, admitting a suitable selector, together with a summability condition of the form

$$\mathbb{E} \exp(\beta A_T) + \mathbb{E}[|g(X_T^{t,x})|^2 e^{\beta A_T}] < \infty$$

for a sufficiently large value of β, we prove that the optimal control problem has a solution, and that the value function and the optimal control can be represented by means of the solution to the BSDE.

We note that optimal control of point processes is a classical topic in stochastic analysis, and the first main contributions date back several decades: we refer the reader for instance to the corresponding chapters of the treatises [6] and [15]. The markovian case has been further investigated in depth, even for more general classes of processes, see e.g. [12]. The results we present in this paper are an attempt toward an alternative systematic approach, based on BSDEs. We hope this may lead to useful results in the future, for instance in connection with computational issues and a better understanding of the nonmarkovian situation. Although this approach is analogous to the diffusive case, it seems that it is pursued here for the first time in the case of marked point processes. In particular it differs from the control-theoretic applications addressed in [24], devoted to a version of the stochastic maximum principle.

Finally, in section 5, we introduce the following Hamilton-Jacobi-Bellman equation (HJB for short) associated to the optimal control problem described above:

$$\begin{aligned} v(t, x) &+ \int_t^T \int_K V(s, x, y)\, q(ds\, dy) \\ &= g(x) + \int_t^T \int_K \Big(v(s, y) - v(s, x) + V(s, y, y) - V(s, x, y) \Big)\, \phi_s(dy)\, dA_s \\ &+ \int_t^T f\Big(s, x, v(s, \cdot) - v(s, x) + V(s, \cdot, \cdot)\Big)\, dA_s, \qquad t \in [0, T],\ x \in K, \end{aligned} \tag{1.4}$$

where f be the hamiltonian function defined in (1.3). The solution is a pair of random fields $\{v(t,x), V(t,x,y) \,:\, t \in [0,T],\, x,y \in K\}$, and in this non-markovian framework the HJB equation is stochastic and of backward type, driven by the same random measure as before. Thus, the previous results are applied to prove its well-posedness. For technical reasons, however, we limit ourselves to the case where the state space K is at most countable: although this is a considerable restriction with respect to the previous results, it allows to treat important classes of control problems, for instance those related to queuing systems. Under appropriate assumptions, similar to those outlined above, we prove that the HJB equation is well-posed and that $v(t,x)$ coincide with the (stochastic) value function of the optimal control problem and it can be represented by means of the associated BSDE. A backward stochastic HJB equation has been first introduced in [21] in the diffusive case, where the corresponding theory is still not complete due to greater technical difficulties. It is an interesting fact that the parallel case of jump processes can be treated using BSDEs and fairly complete results can be given, at least under the restriction mentioned above: this is perhaps due to the different nature of the control problem (here the laws of the controlled processes are obtained via an absolutely continuous change of measure, in contrast to [21]). We borrow some ideas from [21], in particular the use of a formula of Ito-Kunita type proven below, that suggested the unusual form of (1.4). We are not aware of any previous result on backward HJB equations in a non-diffusive context.

The results of this paper admit several variants and generalizations: some of them are not included here for reasons of brevity and some are presently in preparation. For instance, the BSDE approach to optimal control of Markov jump processes deserves a specific treatment; moreover, BSDEs driven by random measures can be studied without Lipschitz assumptions on the generator, along the lines of the many results available in the diffusive case, or extensions to the case of vector-valued process Y or of random time interval can be considered.

2 Notations, preliminaries and basic assumptions

In this section we are going to recall basic notions on marked points processes, random measures and corresponding stochastic integrals, that will be constantly used in the rest of the paper. We also formulate several assumptions that will remain in force throughout.

2.1 Marked point processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(K, \mathcal{K})$ a measurable space. Assume we have a sequence $(T_n, \xi_n)_{n\geq 1}$ of random variables, T_n taking values in $[0,\infty]$ and ξ_n in K. We set $T_0 = 0$ and we assume, $\mathbb{P}$-a.s.,

$$T_n < \infty \implies T_n < T_{n+1}, \qquad n \geq 0.$$

We call (T_n) a point process and (T_n, ξ_n) a marked point process. K is called the mark space, or state space.

In this paper we will always assume that (T_n) is nonexplosive, i.e. $T_n \to \infty$ $\mathbb{P}$-a.s.

For every $A \in \mathcal{K}$ we define the counting processes

$$N_t(A) = \sum_{n\geq 1} 1_{T_n \leq t} 1_{\xi_n \in A}, \qquad t \geq 0$$

and we set $N_t = N_t(K)$. We define the filtration generated by the counting processes by first introducing the σ-algebras

$$\mathcal{F}_t^0 = \sigma(N_s(A) \,:\, s \in [0,t],\, A \in \mathcal{K}), \qquad t \geq 0,$$

and setting

$$\mathcal{F}_t = \sigma(\mathcal{F}_t^0, \mathcal{N}), \qquad t \geq 0,$$

where $\mathcal{N}$ denotes the family of $\mathbb{P}$-null sets in $\mathcal{F}$. It turns out that $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and therefore satisfies the usual conditions. In the following all measurability concepts for stochastic processes (e.g. adaptedness, predictability) will refer to the filtration $(\mathcal{F}_t)_{t\geq 0}$. The predictable σ-algebra (respectively, the progressive σ-algebra) on $\Omega \times [0, \infty)$ will be denoted by $\mathcal{P}$ (respectively, by $Prog$). The same symbols will also denote the restriction to $\Omega \times [0, T]$ for some $T > 0$.

It is known that there exists an increasing, right-continuous predictable process A satisfying $A_0 = 0$ and

$$\mathbb{E} \int_0^\infty H_t \, dN_t = \mathbb{E} \int_0^\infty H_t \, dA_t$$

for every nonnegative predictable process H. The above stochastic integrals are defined for $\mathbb{P}$-almost every ω as ordinary (Stieltjes) integrals. A is called the compensator, or the dual predictable projection, of N. In the following we will always make the basic assumption that $\mathbb{P}$-a.s.

$$A \text{ has continuous trajectories} \tag{2.1}$$

which are in particular finite-valued.

We finally fix $\xi_0 \in K$ (deterministic) and we define

$$X_t = \sum_{n\geq 0} \xi_n \, 1_{[T_n, T_{n+1})}(t), \qquad t \geq 0. \tag{2.2}$$

We do not assume that $\mathbb{P}(\xi_n \neq \xi_{n+1}) = 1$. Therefore in general trajectories of $(T_n, \xi_n)_{n\geq 0}$ cannot be reconstructed from trajectories of $(X_t)_{t\geq 0}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$ is not the natural completed filtration of $(X_t)_{t\geq 0}$.

2.2 Random measures and their compensators

For $\omega \in \Omega$ we define a measure on $((0, \infty) \times K, \mathcal{B}((0, \infty)) \otimes \mathcal{K})$ setting

$$p(\omega, C) = \sum_{n\geq 1} 1_{(T_n(\omega), \xi_n(\omega)) \in C} \qquad C \in \mathcal{B}((0, \infty)) \otimes \mathcal{K},$$

where $\mathcal{B}(\Lambda)$ denotes the Borel σ-algebra of any topological space Λ. p is called a random measure since $\omega \mapsto p(\omega, C)$ is $\mathcal{F}$-measurable for fixed C. We also use the notation $p(\omega, dt\, dy)$ or $p(dt\, dy)$. Notice that $p((0, t] \times A) = N_t(A)$ for $t > 0, A \in \mathcal{K}$.

Under mild assumptions on K it can be proved that there exists a function $\phi_t(\omega, A)$ such that

1. for every $\omega \in \Omega$, $t \in [0, \infty)$, the mapping $A \mapsto \phi_t(\omega, A)$ is a probability measure on $(K, \mathcal{K})$;
2. for every $A \in \mathcal{K}$, the process $(\omega, t) \mapsto \phi_t(\omega, A)$ is predictable;
3. for every nonnegative $H_t(\omega, y)$, $\mathcal{P} \otimes \mathcal{K}$-measurable, we have

$$\mathbb{E} \int_0^\infty H_t(y) \, p(dt\, dy) = \mathbb{E} \int_0^\infty H_t(y) \, \phi_t(dy) \, dA_t.$$

For instance, this holds if $(K, \mathcal{K})$ is a Lusin space with its Borel σ-algebra (see [17] Section 2), but since the Lusin property will not play any further role below, in the following we will simply assume the existence of $\phi_t(dy)$ satisfying 1-2-3 above.

The random measure $\phi_t(\omega, dy) \, dA_t(\omega)$ will be denoted $\tilde{p}(\omega, dt\, dy)$, or simply $\tilde{p}(dt\, dy)$, and will be called the compensator, or the dual predictable projection, of p.

2.3 Stochastic integrals

Fix $T > 0$, and let $H_t(\omega, y)$ be a $\mathcal{P} \otimes \mathcal{K}$-measurable real function satisfying

$$\int_0^T \int_K |H_t(y)| \, \phi_t(dy) \, dA_t < \infty, \qquad \mathbb{P} - a.s.$$

Then the following stochastic integral can be defined

$$\int_0^t \int_K H_s(y) \, q(ds \, dy) := \int_0^t \int_K H_s(y) \, p(ds \, dy) - \int_0^t \int_K H_s(y) \, \phi_s(dy) \, dA_s, \qquad t \in [0, T], \tag{2.3}$$

as the difference of ordinary integrals with respect to p and $\tilde{p}$. Here and in the following the symbol $\int_a^b$ is to be understood as an integral over the interval $(a, b]$. We shorten this identity writing $q(dt \, dy) = p(dt \, dy) - \tilde{p}(dt \, dy) = p(dt \, dy) - \phi_t(dy) \, dA_t$. Note that

$$\int_0^t \int_K H_s(y) \, p(ds \, dy) = \sum_{n \geq 1, T_n \leq t} H_{T_n}(\xi_n)$$

is always well defined since we are assuming that $T_n \to \infty$ $\mathbb{P}$-a.s.

For $r \geq 1$ we define $\mathcal{L}^{r,0}(p)$ as the space of $\mathcal{P} \otimes \mathcal{K}$-measurable real functions $H_t(\omega, y)$ such that

$$\mathbb{E} \int_0^T \int_K |H_t(y)|^r \, p(dt \, dy) = \mathbb{E} \int_0^T \int_K |H_t(y)|^r \, \phi_t(dy) \, dA_t < \infty$$

(the equality of the integrals follows from the definition of $\phi_t(dy)$). Given an element H of $\mathcal{L}^{1,0}(p)$, the stochastic integral (2.3) turns out to be a a finite variation martingale.

The key result used in the construction of a solution to the BSDE (3.1) is the integral representation theorem of marked point process martingales (see e.g. [11],[12]), which is a counterpart of the well known representation result for Brownian martingales (see e.g. [22] Ch V.3 or [15] Thm 12.33). Recall that $(\mathcal{F}_t)$ is the filtration generated by the jump process, augmented in the usual way.

Theorem 2.1 *Let M be a cadlag $(\mathcal{F}_t)$-martingale on $[0, T]$. Then we have*

$$M_t = M_0 + \int_0^t \int_K H_s(y) \, q(ds \, dy), \qquad t \in [0, T],$$

for some process $H \in \mathcal{L}^{1,0}(p)$.

2.4 A family of marked point processes.

In the following, in order to use dynamic programming arguments, it will be useful to introduce a family of processes instead of the single process X, each starting at a different time from different points.

Let (T_n, ξ_n) be the marked point process introduced in section 2.1. We fix $t \geq 0$ and we introduce counting processes relative to the time interval $[t, \infty)$ setting

$$N_s^t(A) = \sum_{n \geq 1} 1_{t < T_n \leq s} 1_{\xi_n \in A}, \qquad s \in [t, \infty), \; A \in \mathcal{K},$$

and $N_s^t = N_s^t(K)$. Then $N_s^t(A) = p^t((t, s] \times A)$ for $s > t, A \in \mathcal{K}$, where the random measure p^t is the restriction of p to $(t, \infty) \times K$. With these definitions it is easily verified that the compensator

of p^t (respectively, N^t) is the restriction of $\phi_s(dy)\,dA_s$ (respectively, A) to $[t,\infty)\times K$ (respectively, $[t,\infty)$).

Now we fix $t \geq 0$ and $x \in K$. Noting that N_t is the number of jump times T_n in the interval $[0,t]$, so that $T_{N_t} \leq t < T_{N_t+1}$, we define

$$X_s^{t,x} = x\,1_{[t,T_{N_t+1})}(s) + \sum_{n \geq N_t+1} \xi_n\,1_{[T_n,T_{n+1})}(s), \qquad s \in [t,\infty).$$

In particular, recalling the definition of the process X, previously defined by formula (2.2) and starting at point $\xi_0 \in K$, we observe that $X = X^{0,\xi_0}$.

For arbitrary t, x we also have $X_s^{t,x} = X_s$ for $s \geq T_{N_t+1}$ and, finally, for $0 \leq u \leq t \leq s$ and $x \in K$ the identity $X_s^{t,X_t^{u,x}} = X_s^{u,x}$ is easy to verify.

3 The backward equation

From now on, we fix a deterministic terminal time $T > 0$.

For given $\omega \in \Omega$ and $t \in [0,T]$, we denote $\mathcal{L}^r(K,\mathcal{K},\phi_t(\omega,dy))$ the usual space of $\mathcal{K}$-measurable maps $z : K \to \mathbb{R}$ such that $\int_K |z(y)|^r \phi_t(\omega,dy) < \infty$ (below we will only use $r = 0$ or 1).

Next we introduce several classes of stochastic processes, depending on a parameter $\beta > 0$.

- $\mathcal{L}^{2,\beta}_{Prog}(\Omega \times [0,T])$ denotes the set of real progressive processes Y such that
$$|Y|^2_\beta := \mathbb{E}\int_0^T e^{\beta A_t}|Y_t|^2 dA_t < \infty.$$
- $\mathcal{L}^{2,\beta}(p)$ denotes the set of mappings $Z : \Omega \times [0,T] \times K \to \mathbb{R}$ which are $\mathcal{P} \otimes \mathcal{K}$-measurable and such that
$$\|Z\|^2_\beta := \mathbb{E}\int_0^T \int_K e^{\beta A_t}|Z_t(y)|^2 \phi_t(dy) dA_t < \infty.$$

We say that $Y, Y' \in \mathcal{L}^{2,\beta}_{Prog}(\Omega \times [0,T])$ (respectively, $Z, Z' \in \mathcal{L}^{2,\beta}(p)$) are equivalent if they coincide almost everywhere with respect to the measure $dA_t(\omega)\mathbb{P}(d\omega)$ (respectively, the measure $\phi_t(\omega,dy)dA_t(\omega)\mathbb{P}(d\omega)$) and this happens if and only if $|Y - Y'|_\beta = 0$ (respectively, $\|Z - Z'\|_\beta = 0$). We denote $L^{2,\beta}_{Prog}(\Omega \times [0,T])$ (respectively, $L^{2,\beta}(p)$) the corresponding set of equivalence classes, endowed with the norm $|\cdot|_\beta$ (respectively, $\|\cdot\|_\beta$). $L^{2,\beta}_{Prog}(\Omega \times [0,T])$ and $L^{2,\beta}(p)$ are Hilbert spaces, isomorphic to $L^{2,\beta}(\Omega \times [0,T], Prog, e^{\beta A_t(\omega)}\,dA_t(\omega)\,\mathbb{P}(d\omega))$ and $L^{2,\beta}(\Omega \times [0,T] \times K, \mathcal{P} \otimes \mathcal{K}, e^{\beta A_t(\omega)}\,\phi_t(\omega,dy)\,dA_t(\omega)\,\mathbb{P}(d\omega))$ respectively.

Finally we introduce the Hilbert space $\mathbb{K}^\beta = L^{2,\beta}_{Prog}(\Omega \times [0,T]) \times L^{2,\beta}(p)$, endowed with the norm $||(Y,Z)||^2_\beta := |Y|^2_\beta + ||Z||^2_\beta$.

In the following we will consider the backward stochastic differential equation: $\mathbb{P}$-a.s.,

$$Y_t + \int_t^T \int_K Z_s(y)\,q(ds\,dy) = \xi + \int_t^T f_s(Y_s, Z_s(\cdot))\,dA_s, \qquad t \in [0,T], \tag{3.1}$$

where the generator f and the final condition ξ are given and and we look for unknown processes $(Y,Z) \in \mathbb{K}^\beta$.

Let us consider the following assumptions on the data f and ξ.

Hypothesis 3.1 *1. The final condition $\xi : \Omega \to \mathbb{R}$ is $\mathcal{F}_T$-measurable and $\mathbb{E}\,e^{\beta A_T}|\xi|^2 < \infty$.*

2. *For every $\omega \in \Omega$, $t \in [0,T]$, $r \in \mathbb{R}$, a mapping $f_t(\omega, r, \cdot) : \mathcal{L}^2(K, \mathcal{K}, \phi_t(\omega, dy)) \to \mathbb{R}$ is given, satisfying the following assumptions:*

   (i) for every $Z \in \mathcal{L}^{2,\beta}(p)$ the mapping

$$(\omega, t, r) \mapsto f_t(\omega, r, Z_t(\omega, \cdot)) \tag{3.2}$$

   is $Prog \otimes \mathcal{B}(\mathbb{R})$-measurable;

   (ii) there exists $L \geq 0$, $L' \geq 0$ such that for every $\omega \in \Omega$, $t \in [0,T]$, $r, r' \in \mathbb{R}$, $z, z' \in \mathcal{L}^2(K, \mathcal{K}, \phi_t(\omega, dy))$ we have

$$|f_t(\omega, r, z(\cdot)) - f_t(\omega, r', z'(\cdot))| \leq L'|r - r'| + L \left(\int_K |z(y) - z'(y)|^2 \phi_t(\omega, dy) \right)^{1/2}; \tag{3.3}$$

   (iii) We have

$$\mathbb{E} \int_0^T e^{\beta A_t} |f_t(0,0)|^2 dA_t < \infty. \tag{3.4}$$

Remark 3.2 1. The slightly involved measurability condition on the generator seems unavoidable, since the mapping $f_t(\omega, r, \cdot)$ has a domain which depends on (ω, t). However, in the following section, we will see how it can be effectively verified in connection with optimal control problems.

Note that if $Z \in \mathcal{L}^{2,\beta}(p)$ then $Z_t(\omega, \cdot)$ belongs to $\mathcal{L}^2(K, \mathcal{K}, \phi_t(\omega, dy))$ except possibly on a predictable set of points (ω, t) of measure zero with respect to $dA_t(\omega)\mathbb{P}(d\omega)$, so that the requirement on the measurability of the map (3.2) is meaningful.

2. We note the inclusion

$$\mathcal{L}^{2,\beta}(p) \subset \mathcal{L}^{1,0}(p). \tag{3.5}$$

Indeed if $Z \in \mathcal{L}^{2,\beta}(p)$ then the inequality

$$\int_0^T \int_K |Z_t(y)| \ \phi_t(dy)\, dA_t \leq \left(\int_0^T \int_K |Z_t(y)|^2 \ \phi_t(dy)\, e^{\beta A_t} dA_t \right)^{1/2} \left(\int_0^T e^{-\beta A_t} dA_t \right)^{1/2}$$

and the fact that $\int_0^T e^{-\beta A_t} dA_t = \beta^{-1}(1 - e^{-\beta A_T}) \leq \beta^{-1}$ imply that $Z \in \mathcal{L}^{1,0}(p)$.

It follows from (3.5) that the martingale $M_t = \int_0^t \int_K Z_s(y)\, q(ds\ dy)$ is well defined for $Z \in \mathcal{L}^{2,\beta}(p)$ and has cadlag trajectories $\mathbb{P}$-a.s. It is easily checked that M only depends on the equivalence class of Z as defined above.

Lemma 3.3 *Suppose that $f : \Omega \times [0,T] \to \mathbb{R}$ is progressive, $\xi : \Omega \to \mathbb{R}$ is $\mathcal{F}_T$-measurable and*

$$\mathbb{E}\, e^{\beta A_T} |\xi|^2 + \mathbb{E} \int_0^T e^{\beta A_s} |f_s|^2 dA_s < \infty$$

for some $\beta > 0$. Then there exists a unique pair (Y, Z) in $\mathbb{K}^\beta$ solution to the BSDE

$$Y_t + \int_t^T \int_K Z_s(y)\, q(ds\ dy) = \xi + \int_t^T f_s\, dA_s. \tag{3.6}$$

Moreover the following identity holds:

$$\begin{aligned} &\mathbb{E}\, e^{\beta A_t} |Y_t|^2 + \beta\, \mathbb{E} \int_t^T e^{\beta A_s} |Y_s|^2 dA_s + \mathbb{E} \int_t^T \int_K e^{\beta A_s} |Z_s(y)|^2 \phi_s(dy) dA_s \\ &\quad = \mathbb{E}\, e^{\beta A_T} |\xi|^2 + 2\mathbb{E} \int_t^T e^{\beta A_s} Y_s\, f_s dA_s, \qquad t \in [0,T], \end{aligned} \tag{3.7}$$

and and there exist two constants $c_1(\beta) = 4(1+\frac{1}{\beta})$ *and* $c_2(\beta) = \frac{8}{\beta}(1+\frac{1}{\beta})$ *such that*

$$\mathbb{E}\int_0^T e^{\beta A_s}|Y_s|^2 dA_s + \mathbb{E}\int_0^T \int_K e^{\beta A_s}|Z_s(y)|^2 \phi_s(dy) dA_s \le c_1(\beta)\mathbb{E}\, e^{\beta A_T}|\xi|^2 + c_2(\beta)\int_0^T e^{\beta A_s}|f_s|^2\, dA_s. \tag{3.8}$$

Proof. Uniqueness follows immediately using the linearity of (3.6) and taking the conditional expectation given $\mathcal{F}_t$.

Assuming that $(Y, Z) \in \mathbb{K}^\beta$ is a solution, we now prove the identity (3.7). From the Ito formula applied to $e^{\beta A_t}|Y_t|^2$ it follows that

$$d(e^{\beta A_t}|Y_t|^2) = \beta e^{\beta A_t}|Y_t|^2 dA_t + 2e^{\beta A_t} Y_{t-} dY_t + e^{\beta A_t}|\Delta Y_t|^2.$$

So integrating on $[t, T]$ and recalling that A is continuous,

$$\begin{aligned} e^{\beta A_t}|Y_t|^2 &= -\int_t^T \beta e^{\beta A_s}|Y_s|^2 dA_s - 2\int_t^T e^{\beta A_s} Y_{s-} \int_K Z_s(y) q(ds\, dy) - \sum_{t<s\le T} e^{\beta A_s}|\Delta Y_s|^2 \\ &\quad + e^{\beta A_T}|\xi|^2 + 2\int_t^T e^{\beta A_s} Y_s\, f_s\, dA_s. \end{aligned} \tag{3.9}$$

The integral process $\int_0^t e^{\beta A_s} Y_{s-} \int_K Z_s(y) q(ds\, dy)$ is a martingale, because the integrand process $e^{\beta A_s} Y_{s-} Z_s(y)$ is in $\mathcal{L}^1(p)$: in fact from the Young inequality we get

$$\begin{aligned} &\mathbb{E}\int_0^T \int_K e^{\beta A_s}|Y_{s-}||Z_s(y)|\phi_s(dy) dA_s \\ &\qquad \le \frac{1}{2}\mathbb{E}\int_0^T e^{\beta A_s}|Y_{s-}|^2 dA_s \quad + \quad \frac{1}{2}\mathbb{E}\int_0^T \int_K e^{\beta A_s}|Z_s(y)|^2 \phi_s(dy) dA_s < +\infty. \end{aligned}$$

Moreover we have

$$\begin{aligned} \sum_{0<s\le t} e^{\beta A_s}|\Delta Y_s|^2 &= \int_0^t \int_K e^{\beta A_s}|Z_s(y)|^2 p(ds\, dy) \\ &= \int_0^t \int_K e^{\beta A_s}|Z_s(y)|^2 q(ds\, dy) + \int_0^t \int_K e^{\beta A_s}|Z_s(y)|^2 \phi_s(dy) dA_s, \end{aligned}$$

where the stochastic integral with respect to q is a martingale. Taking the expectation in (3.15) we obtain (3.7).

We now pass to the proof of existence of the required solution. We start from the inequality

$$\int_t^T |f_s|\, dA_s = \int_t^T e^{-\frac{\beta}{2}A_s} e^{\frac{\beta}{2}A_s}|f_s|\, dA_s \le \left(\int_t^T e^{-\beta A_s} dA_s\right)^{1/2} \left(\int_t^T e^{\beta A_s}|f_s|^2 dA_s\right)^{1/2}.$$

Since $\beta\int_t^T e^{-\beta A_s} dA_s = e^{-\beta A_t} - e^{-\beta A_T} \le e^{-\beta A_t}$ we arrive at

$$\left(\int_t^T |f_s|\, dA_s\right)^2 \le \frac{e^{-\beta A_t}}{\beta}\int_t^T e^{\beta A_s}|f_s|^2 dA_s. \tag{3.10}$$

This implies in particular that $\int_0^T |f_s|\, dA_s$ is square summable. The solution (Y, Z) is defined by considering a cadlag version of the martingale $M_t = \mathbb{E}^{\mathcal{F}_t}[\xi + \int_0^T f_s\, dA_s]$. By the martingale representation Theorem 2.1, there exists a process $Z \in \mathcal{L}^{1,0}(p)$ such that

$$M_t = M_0 + \int_0^t \int_K Z_s(y)\; q(dy\, ds), \qquad t \in [0, T].$$

Define the process Y by

$$Y_t = M_t - \int_0^t f_s(U_s, V_s)\, dA_s, \qquad t \in [0,T].$$

Noting that $Y_T = \xi$, we easily deduce that the equation (3.6) is satisfied.

It remains to show that $(Y,Z) \in \mathbb{K}^\beta$. Taking the conditional expectation, it follows from (3.6) that $Y_t = \mathbb{E}^{\mathcal{F}_t}[\xi + \int_t^T f_s\, dA_s]$ so that, using (3.10), we obtain

$$\begin{aligned} e^{\beta A_t}|Y_t|^2 &\le 2e^{\beta A_t}|\mathbb{E}^{\mathcal{F}_t}\xi|^2 + 2e^{\beta A_t}\left|\mathbb{E}^{\mathcal{F}_t}\int_t^T f_s\, dA_s\right|^2 \\ &\le 2\mathbb{E}^{\mathcal{F}_t}\left[e^{\beta A_T}|\xi|^2 + \frac{1}{\beta}\int_0^T e^{\beta A_s}|f_s|^2\, dA_s\right]. \end{aligned} \tag{3.11}$$

Denoting by m_t the right-hand side of (3.11), we see that m is a martingale by the assumptions of the lemma. In particular, for every stopping time S with values in $[0,T]$, we have

$$\mathbb{E}\, e^{\beta A_S}|Y_S|^2 \le \mathbb{E}\, m_S = \mathbb{E}\, m_T < \infty \tag{3.12}$$

by the optional stopping theorem. Next we define the increasing sequence of stopping times

$$S_n = \inf\{t \in [0,T] \,:\, \int_0^t e^{\beta A_s}|Y_s|^2 dA_s + \int_0^t \int_K e^{\beta A_s}|Z_s(y)|^2 \phi_s(dy) dA_s > n\},$$

with the convention $\inf \emptyset = T$. Computing the Ito differential $d(e^{\beta A_t}|Y_t|^2)$ on the interval $[0, S_n]$ and proceeding as before we deduce

$$\begin{aligned} &\beta\, \mathbb{E}\int_0^{S_n} e^{\beta A_s}|Y_s|^2 dA_s + \mathbb{E}\int_0^{S_n}\int_K e^{\beta A_s}|Z_s(y)|^2\phi_s(dy)dA_s \\ &\quad \le \mathbb{E}\, e^{\beta A_{S_n}}|Y_{S_n}|^2 + 2\mathbb{E}\int_0^{S_n} e^{\beta A_s} Y_s f_s\, dA_s. \end{aligned}$$

Using the inequalities $2Y_s f_s \le (\beta/2)|Y_s|^2 + (2/\beta)|f_s|^2$ and (3.12) (with $S = S_n$) we find the following estimates

$$\mathbb{E}\int_0^{S_n} e^{\beta A_s}|Y_s|^2 dA_s \le \frac{4}{\beta}\,\mathbb{E}e^{\beta A_T}|\xi|^2 + \frac{8}{\beta^2}\mathbb{E}\int_0^T e^{\beta A_s}|f_s|^2\, dA_s,$$

$$\mathbb{E}\int_0^{S_n}\int_K e^{\beta A_s}|Z_s(y)|^2\phi_s(dy)dA_s \le 4\mathbb{E}e^{\beta A_T}|\xi|^2 + \frac{8}{\beta}\mathbb{E}\int_0^T e^{\beta A_s}|f_s|^2\, dA_s,$$

from which we deduce

$$\begin{aligned} &\mathbb{E}\int_0^{S_n} e^{\beta A_s}|Y_s|^2 dA_s + \mathbb{E}\int_0^{S_n}\int_K e^{\beta A_s}|Z_s(y)|^2\phi_s(dy)dA_s \\ &\quad \le c_1(\beta)\mathbb{E}e^{\beta A_T}|\xi|^2 + c_2(\beta)\int_0^T e^{\beta A_s}|f_s|^2\, dA_s, \end{aligned} \tag{3.13}$$

where $c_1(\beta) = 4(1 + \frac{1}{\beta})$ and $c_2(\beta) = \frac{8}{\beta}(1 + \frac{1}{\beta})$.

Setting $S = \lim_n S_n$ we deduce

$$\int_0^S e^{\beta A_s}|Y_s|^2 dA_s + \int_0^S \int_K e^{\beta A_s}|Z_s(y)|^2\phi_s(dy)dA_s < \infty, \qquad \mathbb{P}-a.s.$$

which implies $S = T$, $\mathbb{P}$-a.s., by the definition of S_n. Letting $n \to \infty$ in (3.13) we conclude that (3.8) holds, so that $(Y,Z) \in \mathbb{K}^\beta$. □

Theorem 3.4 *Suppose that Hypothesis 3.1 holds with $\beta > L^2 + 2L'$.*

Then there exists a unique pair (Y, Z) in $\mathbb{K}^\beta$ which solves the BSDE (3.1).

Proof. We use a fixed point theorem for the mapping $\Gamma : \mathbb{K}^\beta \to \mathbb{K}^\beta$ defined setting $(Y, Z) = \Gamma(U, V)$ if (Y, Z) is the pair satisfying

$$Y_t + \int_t^T \int_K Z_s(y)\, q(ds\, dy) = \xi + \int_t^T f_s(U_s, V_s)\, dA_s. \tag{3.14}$$

Let us remark that from the assumptions on f it follows that $\mathbb{E} \int_0^T e^{\beta A_s} |f_s(U_s, V_s)|^2 dA_s < \infty$, so by Lemma 3.3 there exists a unique $(Y, Z) \in \mathbb{K}^\beta$ satisfying (3.14) and Γ is a well defined map.

Let (U^i, V^i), $i = 1, 2$, be elements of $\mathbb{K}^\beta$ and let $(Y^i, Z^i) = \Gamma(U^i, V^i)$. Denote $\overline{Y} = Y^1 - Y^2$, $\overline{Z} = Z^1 - Z^2$, $\overline{U} = U^1 - U^2$, $\overline{V} = V^1 - V^2$, $\overline{f}_s = f_s(U_s^1, V_s^1) - f_s(U_s^2, V_s^2)$. Lemma 3.3 applies to $\overline{Y}, \overline{Z}, \overline{f}$ and (3.7) yields, noting that $\overline{Y}_T = 0$,

$$\begin{aligned}
&\mathbb{E} e^{\beta A_t} |\overline{Y}_t|^2 + \beta\, \mathbb{E} \int_t^T e^{\beta A_s} |\overline{Y}_s|^2 dA_s + \mathbb{E} \int_t^T \int_K e^{\beta A_s} |\overline{Z}_s(y)|^2 \phi_s(dy) dA_s \\
&\quad = 2\mathbb{E} \int_t^T e^{\beta A_s} \overline{Y}_s\, \overline{f}_s dA_s, \qquad t \in [0, T],
\end{aligned}$$

From the Lipschitz conditions of f and elementary inequalities it follows that

$$\begin{aligned}
&\beta\, \mathbb{E} \int_0^T e^{\beta A_s} |\overline{Y}_s|^2 dA_s + \mathbb{E} \int_0^T \int_K e^{\beta A_s} |\overline{Z}_s(y)|^2 \phi_s(dy) dA_s \\
&\quad \le 2L\mathbb{E} \int_t^T e^{\beta A_s} |\overline{Y}_s| \left(\int_K |\overline{V}_s(y)|^2 \phi_s(dy) \right)^{1/2} dA_s + 2L'\mathbb{E} \int_t^T e^{\beta A_s} |\overline{Y}_s|\, |\overline{U}_s|\, dA_s \\
&\quad \le \alpha \mathbb{E} \int_0^T \int_K e^{\beta A_s} |\overline{V}_s(y)|^2 \phi_s(dy)\, dA_s + \frac{L^2}{\alpha} \mathbb{E} \int_0^T e^{\beta A_s} |\overline{Y}_s|^2\, dA_s \\
&\qquad + \gamma L' \mathbb{E} \int_0^T e^{\beta A_s} |\overline{Y}_s|^2\, dA_s + \frac{L'}{\gamma} \mathbb{E} \int_0^T e^{\beta A_s} |\overline{U}_s|^2\, dA_s
\end{aligned}$$

for every $\alpha > 0$, $\gamma > 0$. This can be written

$$\left(\beta - \frac{L^2}{\alpha} - \gamma L' \right) |\overline{Y}|_\beta^2 + \|\overline{Z}\|_\beta^2 \le \alpha \|\overline{V}\|_\beta^2 + \frac{L'}{\gamma} |\overline{U}|_\beta^2.$$

By the assumption on β it is possible to find $\alpha \in (0, 1)$ such that

$$\beta > \frac{L^2}{\alpha} + \frac{2L'}{\sqrt{\alpha}}.$$

If $L' = 0$ we see that Γ is an α-contraction on $\mathbb{K}^\beta$ endowed with the equivalent norm $(Y, Z) \mapsto (\beta - (L^2/\alpha))\, |Y|_\beta^2 + \|Z\|_\beta^2$. If $L' > 0$ we choose $\gamma = 1/\sqrt{\alpha}$ and obtain

$$\frac{L'}{\sqrt{\alpha}} |\overline{Y}|_\beta^2 + \|\overline{Z}\|_\beta^2 \le \alpha \|\overline{V}\|_\beta^2 + L' \sqrt{\alpha} |\overline{U}|_\beta^2 = \alpha \left(\frac{L'}{\sqrt{\alpha}} |\overline{U}|_\beta^2 + \|\overline{V}\|_\beta^2 \right),$$

so that Γ is an α-contraction on $\mathbb{K}^\beta$ endowed with the equivalent norm $(Y, Z) \mapsto (L'/\sqrt{\alpha})\, |Y|_\beta^2 + \|Z\|_\beta^2$. In all cases there exists a unique fixed point which is the required unique solution to the BSDE (3.1). □

We next prove some estimates on the solutions of the BSDE, which show in particular the continuous dependence upon the data. Let us consider two solutions (Y^1, Z^1), $(Y^2, Z^2) \in \mathbb{K}^\beta$ to the BSDE (3.1) associated with the drivers f^1 and f^2 and final data ξ^1 and ξ^2, respectively, which are assumed to satisfy Hypothesis 3.1. Denote $\overline{Y} = Y^1 - Y^2$, $\overline{Z} = Z^1 - Z^2$, $\overline{\xi} = \xi^1 - \xi^2$, $\overline{f}_s = f_s^1(Y_s^2, Z_s^2(\cdot)) - f_s^2(Y_s^2, Z_s^2(\cdot))$.

Proposition 3.5 *Let $(\overline{Y}, \overline{Z})$ be the processes defined above. Then, for $\beta > 2L' + L^2$, the a priori estimates hold:*

$$
\begin{aligned}
|\overline{Y}|^2_\beta &\leq \frac{2}{\beta - 2L' - L^2}\, \mathbb{E} e^{\beta A_T} |\overline{\xi}|^2 + \frac{4}{(\beta - 2L' - L^2)^2}\, \mathbb{E} \int_0^T e^{\beta A_s} |\overline{f}_s|^2 dA_s, \\
\|\overline{Z}\|^2_\beta &\leq \left(2 + \frac{16}{\beta - 2L' - L^2}\right) \mathbb{E} e^{\beta A_T} |\overline{\xi}|^2 \\
&\quad + \frac{2}{\beta - 2L' - L^2} \left(1 + \frac{16}{\beta - 2L' - L^2}\right) \mathbb{E} \int_0^T e^{\beta A_s} |\overline{f}_s|^2 dA_s.
\end{aligned}
$$

Proof. From the Ito formula applied to $e^{\beta A_t} |\overline{Y}_t|^2$ it follows that

$$
d(e^{\beta A_t} |\overline{Y}_t|^2) = \beta e^{\beta A_t} |\overline{Y}_t|^2 dA_t + 2 e^{\beta A_t} \overline{Y}_{t-} dY_t + e^{\beta A_t} |\Delta \overline{Y}_t|^2.
$$

So integrating on $[t, T]$ and recalling that A is continuous,

$$
\begin{aligned}
e^{\beta A_t} |\overline{Y}_t|^2 &= -\int_t^T \beta e^{\beta A_s} |\overline{Y}_s|^2 dA_s - 2 \int_t^T e^{\beta A_s} \overline{Y}_{s-} \int_K \overline{Z}_s(y) q(ds\, dy) - \sum_{t < s \leq T} e^{\beta A_s} |\Delta \overline{Y}_s|^2 \\
&\quad + e^{\beta A_T} |\overline{\xi}|^2 + 2 \int_t^T e^{\beta A_s} \overline{Y}_s (f^1(Y^1_s, Z^1_s(\cdot)) - f^2(Y^2_s, Z^2_s(\cdot)) dA_s. \qquad (3.15)
\end{aligned}
$$

The integral process $\int_0^t e^{\beta A_s} \overline{Y}_{s-} \int_K Z_s(y) q(ds\, dy)$ is a martingale, because the integrand process $e^{\beta A_s} \overline{Y}_{s-} \overline{Z}_s(y)$ is in $L^1(p)$: in fact from the Young inequality we get

$$
\begin{aligned}
&\mathbb{E} \int_0^T \int_K e^{\beta A_s} |\overline{Y}_{s-}| |\overline{Z}_s(y)| \phi_s(dy) dA_s \\
&\qquad \leq \frac{1}{2} \mathbb{E} \int_0^T e^{\beta A_s} |\overline{Y}_{s-}|^2 dA_s \quad + \quad \frac{1}{2} \mathbb{E} \int_0^T \int_K e^{\beta A_s} |\overline{Z}_s(y)|^2 \phi_s(dy) dA_s < +\infty.
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
\sum_{0 < s \leq t} e^{\beta A_s} |\Delta \overline{Y}_s|^2 &= \int_0^t \int_K e^{\beta A_s} |\overline{Z}_s(y)|^2 p(ds\, dy) \\
&= \int_0^t \int_K e^{\beta A_s} |\overline{Z}_s(y)|^2 q(ds\, dy) + \int_0^t \int_K e^{\beta A_s} |\overline{Z}_s(y)|^2 \phi_s(dy) dA_s,
\end{aligned}
$$

where the stochastic integral with respect to q is a martingale.

Hence taking the expectation in (3.15), by the Lipschitz property of the driver f^1 and using the notation $\|z(\cdot)\|^2_s = \int_K |z(y)|^2 \phi_s(dy)$ we get

$$
\begin{aligned}
\mathbb{E} e^{\beta A_t} |\overline{Y}_t|^2 &= -\mathbb{E} \int_t^T \beta e^{\beta A_s} |\overline{Y}_s|^2 dA_s - \mathbb{E} \int_t^T \int_K e^{\beta A_s} |\overline{Z}_s(y)|^2 \phi_s(dy) dA_s + \mathbb{E} e^{\beta A_T} |\overline{\xi}|^2 \\
&\quad + 2\mathbb{E} \int_t^T e^{\beta A_s} \overline{Y}_s (f^1(Y^1_s, Z^1_s) - f^2(Y^2_s, Z^2_s) dA_s \\
&\leq -\mathbb{E} \int_t^T \beta e^{\beta A_s} |\overline{Y}_s|^2 dA_s - \mathbb{E} \int_t^T \int_K e^{\beta A_s} |\overline{Z}_s(y)|^2 \phi_s(dy) dA_s + \mathbb{E} e^{\beta A_T} |\overline{\xi}|^2 \\
&\quad + 2\mathbb{E} \int_t^T e^{\beta A_s} |\overline{Y}_s| (|f^1(Y^1_s, Z^1_s) - f^1(Y^2_s, Z^2_s)| + |\overline{f}_s|) dA_s \\
&\leq -\mathbb{E} \int_t^T \beta e^{\beta A_s} |\overline{Y}_s|^2 dA_s - \mathbb{E} \int_t^T e^{\beta A_s} \|\overline{Z}_s\|^2_s dA_s + \mathbb{E} e^{\beta A_T} |\overline{\xi}|^2 \\
&\quad + 2L' \mathbb{E} \int_t^T e^{\beta A_s} |\overline{Y}_s|^2 dA_s + 2L \mathbb{E} \int_t^T e^{\beta A_s} |\overline{Y}_s| \|\overline{Z}_s\|_s dA_s + 2\mathbb{E} \int_t^T e^{\beta A_s} |\overline{Y}_s| |\overline{f}_s| dA_s.
\end{aligned}
$$

We note that the quantity $Q(y,z) = -\beta|y|^2 - \|z\|_s^2 + 2L'|y|^2 + 2L|y|\|z\|_s + 2|\overline{f}_s||y|$ which occurs in the integrand terms in the right hand of the above inequality can be written as

$$
\begin{aligned}
Q(y,z) &= -\beta|y|^2 + 2L'|y|^2 + L^2|y|^2 + 2|\overline{f}_s||y| - (\|z\|_s - L|y|)^2 \\
&= -\beta_L(|y| - \beta_L^{-1}|\overline{f}_s|)^2 - (\|z\|_s - L|y|)^2 + \beta_L^{-1}|\overline{f}_s|^2
\end{aligned}
$$

where $\beta_L := \beta - 2L' - L^2$ is assumed to be strictly positive. Hence

$$
\begin{aligned}
\mathbb{E}e^{\beta A_t}|\overline{Y}_t|^2 &+ \beta_L \mathbb{E}\int_t^T e^{\beta A_s}(|\overline{Y}_s| - \beta_L^{-1}|\overline{f}_s|)^2 dA_s + \mathbb{E}\int_t^T e^{\beta A_s}(\|\overline{Z}_s\|_s - L|\overline{Y}_s|)^2 dA_s \\
&\leq \mathbb{E}e^{\beta A_T}|\overline{\xi}|^2 + \mathbb{E}\int_t^T e^{\beta A_s}\frac{|\overline{f}_s|^2}{\beta_L} dA_s
\end{aligned}
$$

from which we deduce

$$
\mathbb{E}\int_0^T e^{\beta A_s}|\overline{Y}_s|^2 dA_s \leq \frac{2}{\beta_L}\mathbb{E}e^{\beta A_T}|\overline{\xi}|^2 + \frac{4}{\beta_L^2}\mathbb{E}\int_0^T e^{\beta A_s}|\overline{f}_s|^2 dA_s
$$

and

$$
\mathbb{E}\int_0^T e^{\beta A_s}\|\overline{Z}_s\|_s^2 dA_s \leq \left(2 + \frac{16}{\beta_L}\right)\mathbb{E}e^{\beta A_T}|\overline{\xi}|^2 + \frac{2}{\beta_L}\left(1 + \frac{16}{\beta_L}\right)\mathbb{E}\int_0^T e^{\beta A_s}\frac{|\overline{f}_s|^2}{\beta_L} dA_s
$$

□

From the a priori estimates one can deduce the continuous dependence of the solution upon the data.

Proposition 3.6 *Suppose that Hypothesis 3.1 holds with $\beta > L^2 + 2L'$ and let (Y,Z) be the unique solution in $\mathbb{K}^\beta$ to the BSDE (3.1). Then*

$$
\begin{aligned}
\mathbb{E}\int_0^T e^{\beta A_s}|Y_s|^2 dA_s &+ \mathbb{E}\int_0^T e^{\beta A_s}\int_K |Z_s(y)|^2 \phi_s(dy) dA_s \\
&\leq C_1(\beta)\mathbb{E}e^{\beta A_T}|\xi|^2 + C_2(\beta)\int_0^T e^{\beta A_s}\frac{|\overline{f}_s|^2}{\beta_L} dA_s,
\end{aligned}
$$

where $C_1(\beta) = \left(2 + \frac{18}{\beta - 2L' - L^2}\right)$, $C_2(\beta) = \frac{2}{\beta - 2L' - L^2}\left(1 + \frac{18}{\beta - 2L' - L^2}\right)$.

Proof. The thesis follows from Proposition 3.5 setting $f^1 = f$, $\xi^1 = \xi$, $f^2 = 0$ and $\xi^2 = 0$. □

4 Optimal control

Throughout this section we assume that a marked point process is given, satisfying the assumptions of Section 2. In particular we suppose that $T_n \to \infty$ $\mathbb{P}$-a.s. and that (2.1) holds.

The data specifying the optimal control problem are an action (or decision) space U, a running cost function l, a terminal cost function g, and another function r specifying the effect of the control process. They are assumed to satisfy the following conditions.

Hypothesis 4.1 *1. $(U, \mathcal{U})$ is a measurable space.*

2. The functions $r, l : \Omega \times [0,T] \times K \times U \to \mathbb{R}$ are $\mathcal{P} \otimes \mathcal{K} \otimes \mathcal{U}$-measurable and there exist constants $C_r > 1$, $C_l > 0$ such that, $\mathbb{P}$-a.s.,

$$
0 \leq r_t(y,u) \leq C_r, \quad |l_t(x,u)| \leq C_l, \qquad t \in [0,T], x \in K, u \in U. \tag{4.1}
$$

3. The function $g : \Omega \times K \to \mathbb{R}$ is $\mathcal{F}_T \otimes \mathcal{K}$-measurable.

We define as an admissible control process, or simply a control, any predictable process $(u_t)_{t\in[0,T]}$ with values in U. The set of admissible control processes is denoted $\mathcal{A}$.

To every control $u(\cdot) \in \mathcal{A}$ we associate a probability measure $\mathbb{P}_u$ on $(\Omega, \mathcal{F})$ by a change of measure of Girsanov type, as we now describe. We define

$$L_t = \exp\left(\int_0^t \int_K (1 - r_s(y, u_s))\, \phi_s(dy)\, dA_s\right) \prod_{n\geq 1\,:\, T_n\leq t} r_{T_n}(\xi_n, u_{T_n}), \qquad t \in [0, T],$$

with the convention that the last product equals 1 if there are no indices $n \geq 1$ satisfying $T_n \leq t$ (similar conventions will be adopted later without further mention). It is a well-known result that L is a nonnegative supermartingale, (see [17] Proposition 4.3, or [4]), solution to the equation

$$L_t = 1 + \int_0^t \int_K L_{s-}\,(r_s(y, u_s) - 1)\; q(ds\,dy), \qquad t \in [0, T].$$

The following result collects some properties of the process L that we need later.

Lemma 4.2 *Let $\gamma > 1$ and*

$$\beta = \gamma + 1 + \frac{C_r^{\gamma^2}}{\gamma - 1}. \tag{4.2}$$

If $\mathbb{E}\exp(\beta A_T) < \infty$ then we have $\sup_{t\in[0,T]} \mathbb{E}L_t^\gamma < \infty$ and $\mathbb{E}L_T = 1$.

Proof. We follow [6], Chapter VIII Theorem T11, with some modifications. To shorten notation we define $\rho_s(y) = r_s(y, u_s)$ and we denote $L_t = \mathcal{E}(\rho)_t$. For $\gamma > 1$ we define

$$a_s(y) = \gamma^{-1}(1 - \rho_s(y)^{\gamma^2}), \qquad b_s(y) = \gamma - \gamma\rho_s(y) - \gamma^{-1} + \gamma^{-1}\rho_s(y)^{\gamma^2},$$

so that $\gamma(1 - \rho_s(y)) = a_s(y) + b_s(y)$. Then

$$L_t^\gamma = \exp\left(\int_0^t \int_K (a_s(y) + b_s(y))\, \phi_s(dy)\, dA_s\right) \prod_{T_n\leq t} \rho_{T_n}(\xi_n)^\gamma,$$

and by Hölder's inequality

$$\begin{aligned}\mathbb{E}L_t^\gamma \quad\leq\quad & \left\{\mathbb{E}\Big[\exp\left(\int_0^t \int_K \gamma a_s(y)\, \phi_s(dy)\, dA_s\right) \prod_{T_n\leq t} \rho_{T_n}(\xi_n)^{\gamma^2}\Big]\right\}^{\frac{1}{\gamma}} \\ & \left\{\mathbb{E}\exp\left(\int_0^t \int_K \frac{\gamma}{\gamma-1} b_s(y)\; \phi_s(dy)\, dA_s\right)\right\}^{\frac{\gamma-1}{\gamma}}.\end{aligned}$$

Noting that $\gamma a_s(y) = 1 - \rho_s(y)^{\gamma^2}$, the term in square brackets equals $\mathcal{E}(\rho^{\gamma^2})_t$ and we have $\mathbb{E}\mathcal{E}(\rho^{\gamma^2})_t \leq 1$ by the supermartingale property. Since $b_s(y) \leq \gamma - \gamma^{-1} + \gamma^{-1}C_r^{\gamma^2}$ we arrive at

$$\mathbb{E}L_t^\gamma \leq \left\{\mathbb{E}\exp\left(A_T\left(\gamma + 1 + \frac{C_r^{\gamma^2}}{\gamma - 1}\right)\right)\right\}^{\frac{\gamma-1}{\gamma}} = \left\{\mathbb{E}\exp\left(\beta A_T\right)\right\}^{\frac{\gamma-1}{\gamma}} < \infty. \tag{4.3}$$

Let $S_n = \inf\{t \in [0, T] \,:\, L_{t-} + A_t \geq n\}$ with the convention $\inf\emptyset = T$, and let $\rho_s^{(n)}(y) = 1_{[0,S_n]}(s)\rho_s(y) + 1_{(S_n,T]}(s)$, $L^{(n)} = \mathcal{E}(\rho^{(n)})$. Then $L^{(n)}$ satisfies

$$L_t^{(n)} = 1 + \int_0^t \int_K L_{s-}^{(n)}\,(r_s^{(n)}(y) - 1)\; q(ds\,dy), \qquad t \in [0, T].$$

By the choice of $\rho^{(n)}$ we have $L_t^{(n)} = L_{t\wedge S_n}$, and by the choice of S_n it is easily proved that $\mathbb{E}\int_0^T \int_K L_{s-}^{(n)} |r_s^{(n)}(y) - 1| \, \phi_s(dy) \, dA_s < \infty$, so that $L^{(n)}$ is a martingale and $\mathbb{E}L_t^{(n)} = \mathbb{E}L_{t\wedge S_n} = 1$. The first part of the proof applies to $L^{(n)}$ and the inequality (4.3) yields in particular $\sup_n \mathbb{E}(L_t^{(n)})^\gamma = \sup_n \mathbb{E}(L_{t\wedge S_n})^\gamma < \infty$. So $(L_{t\wedge S_n})_n$ is uniformly integrable and letting $n \to \infty$ we conclude that $\mathbb{E}L_t = 1$. □

Under the assumptions of the lemma, the process L is a martingale and we can define a probability $\mathbb{P}_u$ setting $\mathbb{P}_u(d\omega) = L_T(\omega)\mathbb{P}(d\omega)$. It can then be proved (see [17] Theorem 4.5) that the compensator $\tilde{p}^u$ of p under $\mathbb{P}_u$ is related to the compensator $\tilde{p}$ of p under $\mathbb{P}$ by the formula

$$\tilde{p}^u(dt\,dy) = r_t(y, u_t)\, \tilde{p}(dt\,dy) = r_t(y, u_t)\, \phi_t(dy)\, dA_t.$$

In particular the compensator of N under $\mathbb{P}_u$ is

$$A_t^u = \int_0^t \int_K r_s(y, u_s)\, \phi_s(dy)\, dA_s. \tag{4.4}$$

We finally define the cost associated to every $u(\cdot) \in \mathcal{A}$ as

$$J(u(\cdot)) = \mathbb{E}_u \left[\int_0^T l_t(X_t, u_t)\, dA_t + g(X_T) \right],$$

where $\mathbb{E}_u$ denotes the expectation under $\mathbb{P}_u$. Later we will assume that

$$\mathbb{E}[|g(X_T)|^2 e^{\beta A_T}] < \infty \tag{4.5}$$

for some $\beta > 0$ that will be fixed in such a way that the cost is finite for every admissible control. The control problem consists in minimizing $J(u(\cdot))$ over $\mathcal{A}$.

Remark 4.3 We recall (see e.g. [6], Appendix A2, Theorem T34) that a process u is $(\mathcal{F}_t)$-predictable if and only if it admits the representation

$$u(\omega, t) = \sum_{n\geq 0} u^{(n)}(\omega, t)\, 1_{T_n(\omega) < t \leq T_{n+1}(\omega)} \tag{4.6}$$

where for each $n \geq 0$ the mapping $(\omega, t) \mapsto u^{(n)}(\omega, t)$ is $\mathcal{F}_{T_n} \otimes \mathcal{B}([0, \infty))$-measurable. Since we have $\mathcal{F}_{T_n} = \sigma(T_i, \xi_i, 0 \leq i \leq n)$ (see e.g. [6], Appendix A2, Theorem T30) the fact that a control is predictable can be roughly interpreted by saying that the controller, at each time T_n, based on observation of the random variables $T_i, \xi_i, 0 \leq i \leq n$, chooses his present and future control actions and updates his decisions only at time T_{n+1}.

Remark 4.4 We notice that the laws of the random coefficients r, l, g under $\mathbb{P}$ and under $\mathbb{P}_u$ are not the same in general, so that the formulation of the optimal control problem should be carefully examined when facing a specific application or modeling situation. This difficulty clearly disappears when r, l, g are deterministic.

We next proceed to the solution of the optimal control problem formulated above. A basic role is played by the BSDE

$$Y_t + \int_t^T \int_K Z_s(y)\, q(ds\,dy) = g(X_T) + \int_t^T f(s, X_s, Z_s(\cdot))\, dA_s, \qquad t \in [0, T], \tag{4.7}$$

with terminal condition $g(X_T)$ and generator defined by means of the hamiltonian function f. The hamiltonian function is defined for every $\omega \in \Omega, t \in [0,T], x \in K$ and $z \in \mathcal{L}^1(K, \mathcal{K}, \phi_t(\omega, dy))$ by the formula

$$f(\omega, t, x, z(\cdot)) = \inf_{u \in U} \left\{ l_t(\omega, x, u) + \int_K z(y) \left(r_t(\omega, y, u) - 1 \right) \phi_t(\omega, dy) \right\}. \tag{4.8}$$

We will assume that the infimum is in fact achieved, possibly at many points. Moreover we need to verify that the generator of the BSDE satisfies the conditions required in the previous section. It turns out that an appropriate assumption is the following one, since we will see below (compare Proposition 4.8) that it can be verified under quite general conditions. Here and in the following we set $X_{0-} = X_0$.

Hypothesis 4.5 *For every $Z \in \mathcal{L}^{1,0}(p)$ there exists a function $\underline{u}^Z : \Omega \times [0,T] \to U$, measurable with respect to $\mathcal{P}$ and $\mathcal{U}$, such that*

$$f(\omega, t, X_{t-}(\omega), Z_t(\omega, \cdot)) = l_t(X_{t-}(\omega), \underline{u}^Z(\omega, t)) + \int_K Z_t(\omega, y) \left(r_t(\omega, y, \underline{u}^Z(\omega, t)) - 1 \right) \phi_t(\omega, dy) \tag{4.9}$$

for almost all (ω, t) with respect to the measure $dA_t(\omega)\mathbb{P}(d\omega)$.

Note that if $Z \in \mathcal{L}^{1,0}(p)$ then $Z_t(\omega, \cdot)$ belongs to $\mathcal{L}^1(K, \mathcal{K}, \phi_t(\omega, dy))$ except possibly on a predictable set of points (ω, t) of measure zero with respect to $dA_t(\omega)\mathbb{P}(d\omega)$, so that the equality (4.9) is meaningful. Also note that each u^Z is an admissible control.

We can now verify that all the assumptions of Hypothesis 3.1 hold true for the generator of the BSDE (4.7), which is given by the formula

$$f_t(\omega, z(\cdot)) = f(\omega, t, X_t(\omega), z(\cdot)), \qquad \omega \in \Omega,\ t \in [0,T],\ z \in \mathcal{L}^2(K, \mathcal{K}, \phi_t(\omega, dy)).$$

Indeed, if $Z \in \mathcal{L}^{2,\beta}(p)$ then $Z \in \mathcal{L}^{1,0}(p)$ by (3.5), and (4.9) shows that the process $(\omega, t) \mapsto f(\omega, t, X_{t-}(\omega), Z_t(\omega, \cdot))$ is progressive; since A is assumed to have continuous trajectories and X has piecewise constant pahts, the progressive set $\{(\omega, t) : X_{t-}(\omega) \neq X_t(\omega)\}$ has measure zero with respect to $dA_t(\omega)\mathbb{P}(d\omega)$; it follows that the process

$$(\omega, t) \mapsto f(\omega, t, X_t(\omega), Z_t(\omega, \cdot)) = f_t(\omega, Z_t(\omega, \cdot))$$

is progressive, after modification on a set of measure zero, as required in (3.2). Next, using the boundedness assumptions (4.1), it is easy to check that (3.3) is verified with $L' = 0$ and

$$L = \sup |r - 1| = \sup\{ |r_t(y, u) - 1| \,:\, \omega \in \Omega,\ t \in [0,T],\ y \in K,\ u \in U \}.$$

Using (4.1) again we also have

$$\mathbb{E} \int_0^T e^{\beta A_t} |f(t, X_t, 0)|^2 dA_t = \mathbb{E} \int_0^T e^{\beta A_t} | \inf_{u \in U} l_t(X_t, u)|^2 dA_t \leq C_l^2 \, \beta^{-1} \, \mathbb{E} \, e^{\beta A_T}, \tag{4.10}$$

so that (3.4) holds as well provided the right-hand side of (4.10) is finite. Assuming finally that (4.5) holds, by Theorem 3.4 the BSDE has a unique solution $(Y, Z) \in \mathbb{K}^\beta$ if $\beta > L^2$.

The corresponding admissible control $\underline{u}^Z$, whose existence is required in Hypothesis 4.5, will be denoted u^*.

We are now ready to state the main result of this section. Recall that $C_r > 1$ was introduced in (4.1).

Theorem 4.6 *Assume that Hypotheses 4.1 and 4.5 are satisfied and that*

$$\mathbb{E}\exp\big((3+C_r^4)A_T\big)<\infty. \tag{4.11}$$

Suppose also that there exists β such that

$$\beta>\sup|r-1|^2,\quad \mathbb{E}\exp(\beta A_T)<\infty,\quad \mathbb{E}[|g(X_T)|^2e^{\beta A_T}]<\infty. \tag{4.12}$$

Let $(Y,Z)\in\mathbb{K}^\beta$ denote the solution to the BSDE (4.7) and $u^*=\underline{u}^Z$ the corresponding admissible control. Then $u^*(\cdot)$ is optimal and Y_0 is the optimal cost, i.e. $Y_0=J(u^*(\cdot))=\inf_{u(\cdot)\in\mathcal{A}}J(u(\cdot))$.

Remark 4.7 Note that if g is bounded then (4.12) follows from (4.11) with $\beta=3+C_r^4$, since $|r_t(y,u)-1|^2\le(C_r+1)^2<3+C_r^4$.

Proof. Fix $u(\cdot)\in\mathcal{A}$. Assumption (4.11) allows to apply Lemma 4.2 with $\gamma=2$ and yields $\mathbb{E}L_T^2<\infty$. It follows that $g(X_T)$ is integrable under $\mathbb{P}_u$. Indeed by (4.5)

$$\mathbb{E}_u|g(X_T)|=\mathbb{E}|L_Tg(X_T)|\le(\mathbb{E}L_T^2)^{1/2}(\mathbb{E}g(X_T)^2)^{1/2}<\infty.$$

We next show that under $\mathbb{P}_u$ we have $Z\in\mathcal{L}^{1,0}(p)$, i.e. $\mathbb{E}_u\int_0^T\int_K|Z_t(y)|\,\tilde{p}^u(dt\,dy)<\infty$. First note that, by Hölder's inequality,

$$\begin{aligned}\int_0^T\int_K|Z_t(y)|\,\phi_t(dy)\,dA_t &= \int_0^T\int_K e^{-\frac{\beta}{2}A_t}e^{\frac{\beta}{2}A_t}|Z_t(y)|\,\phi_t(dy)\,dA_t\\ &\le \left(\int_0^T e^{-\beta A_t}dA_t\right)^{1/2}\left(\int_0^T\int_K e^{\beta A_t}|Z_t(y)|^2\,\phi_t(dy)\,dA_t\right)^{1/2}\\ &= \left(\frac{1-e^{-\beta A_T}}{\beta}\right)^{1/2}\left(\int_0^T\int_K e^{\beta A_t}|Z_t(y)|^2\,\phi_t(dy)\,dA_t\right)^{1/2}.\end{aligned}$$

Therefore, using (4.1),

$$\begin{aligned}\mathbb{E}_u\int_0^T\int_K|Z_t(y)|\,\tilde{p}^u(dt\,dy) &= \mathbb{E}_u\int_0^T\int_K|Z_t(y)|\,r_t(y,u_t)\,\phi_t(dy)\,dA_t\\ &= \mathbb{E}\left[L_T\int_0^T\int_K|Z_t(y)|\,r_t(y,u_t)\,\phi_t(dy)\,dA_t\right]\\ &\le (\mathbb{E}L_T^2)^{1/2}\frac{C_r}{\sqrt{\beta}}\left\{\mathbb{E}\int_0^T\int_K e^{\beta A_t}|Z_t(y)|^2\,\phi_t(dy)\,dA_t\right\}^{1/2}\end{aligned}$$

and the right-hand side of the last inequality is finite, since $(Y,Z)\in\mathbb{K}^\beta$. We have now proved that $Z\in\mathcal{L}^{1,0}(p)$ under $\mathbb{P}_u$.

In particular it follows that

$$\mathbb{E}_u\int_0^T\int_K Z_t(y)\,p(dt\,dy)=\mathbb{E}_u\int_0^T\int_K Z_t(y)\,\tilde{p}^u(dt\,dy)=\mathbb{E}_u\int_0^T\int_K Z_t(y)\,r_t(y,u_t)\,\phi_t(dy)\,dA_t.$$

Setting $t=0$ and taking the expectation $\mathbb{E}_u$ in the BSDE (4.7), recalling that $q(dt\,dy)=p(dt\,dy)-\tilde{p}(dt\,dy)=p(dt\,dy)-\phi_t(dy)\,dA_t$ and that Y_0 is deterministic, we obtain

$$Y_0+\mathbb{E}_u\int_0^T\int_K Z_t(y)\,(r_t(y,u_t)-1)\,\phi_t(dy)\,dA_t=\mathbb{E}_u\,g(X_T)+\mathbb{E}_u\int_0^T f(t,X_t,Z_t(\cdot))\,dA_t.$$

We finally obtain

$$\begin{aligned}
Y_0 &= J(u(\cdot)) + \mathbb{E}_u \int_0^T \left[f(t, X_t, Z_t(\cdot)) - l_t(X_t, u_t) - \int_K Z_t(y)\,(r_t(y, u_t) - 1)\,\phi_t(dy) \right] dA_t \\
&= J(u(\cdot)) + \mathbb{E}_u \int_0^T \left[f(t, X_{t-}, Z_t(\cdot)) - l_t(X_{t-}, u_t) - \int_K Z_t(y)\,(r_t(y, u_t) - 1)\,\phi_t(dy) \right] dA_t,
\end{aligned}$$

where the last equality follows from the continuity if A. This identity is sometimes called the fundamental relation. By the definition of the hamiltonian f, the term in square brackets is smaller or equal to 0, and it equals 0 if $u(\cdot) = u^*(\cdot)$. □

Hypothesis 4.5 can be verified in specific situations when it is possible to compute explicitly the functions $\underline{u}^Z$. General conditions for its validity can also be formulated using appropriate selection theorems, as in the following proposition.

Proposition 4.8 *In addition to the assumptions in Hypothesis 4.1, suppose that U is a compact metric space with its Borel σ-algebra $\mathcal{U}$ and that the functions $r_t(\omega, x, \cdot), l_t(\omega, x, \cdot) : U \to \mathbb{R}$ are continuous for every $\omega \in \Omega$, $t \in [0, T]$, $x \in K$. Then Hypothesis 4.5 is verified.*

Proof. Let us consider the measure $\mu(d\omega\, dt) = dA_t(\omega)\mathbb{P}(d\omega)$ on the predictable σ-algebra $\mathcal{P}$. Let $\bar{\mathcal{P}}$ denote its μ-completion and consider the complete measure space $(\Omega \times [0, T], \bar{\mathcal{P}}, \mu)$. Fix $Z \in \mathcal{L}^{1,0}(p)$, note that the set $A^Z = \{(\omega, t) : Z_t(\omega, \cdot) \notin \mathcal{L}^1(K, \mathcal{K}, \phi_t(\omega, dy))$ has μ-measure zero and define a map $F^Z : \Omega \times [0, T] \times U \to \mathbb{R}$ setting

$$F^Z(\omega, t, u) = \begin{cases} l_t(\omega, X_{t-}(\omega), u) + \displaystyle\int_K Z_t(\omega, y) \Big(r_t(\omega, y, u) - 1 \Big) \phi_t(\omega, dy) & \text{if } (\omega, t) \notin A^Z, \\ 0 & \text{if } (\omega, t) \in A^Z. \end{cases}$$

Then $F^Z(\cdot, \cdot, u)$ is $\bar{\mathcal{P}}$-measurable for every $u \in U$, and it is easily verified that $F^Z(\omega, t, \cdot)$ is continuous for every $(\omega, t) \in \Omega \times [0, T]$. By a classical selection theorem (see [1], Theorems 8.1.3 and 8.2.11) there exists a function $\underline{u}^Z : \Omega \times [0, T] \to U$, measurable with respect to $\bar{\mathcal{P}}$ and $\mathcal{U}$, such that $F^Z(\omega, t, \underline{u}^Z(\omega, t)) = \min_{u \in U} F^Z(\omega, t, u)$ for every $(\omega, t) \in \Omega \times [0, T]$, so that (4.9) holds true for every (ω, t). After modification on a set of μ-measure zero, the function u^Z can be made measurable with respect to $\mathcal{P}$ and $\mathcal{U}$, and (4.9) still holds, as it is understood as an equality for μ-almost all (ω, t). □

In several contexts, for instance in order to apply dynamic programming arguments, it is useful to introduce a family of control problems parametrized by $(t, x) \in [0, T] \times K$. Recall the definition of the processes $(X_s^{t,x})_{s \in [t,T]}$ in subsection 2.4.

For fixed (t, x) the cost corresponding to $u \in \mathcal{A}$ is defined as the random variable

$$J_t(x, u(\cdot)) = \mathbb{E}_u^{\mathcal{F}_t} \left[\int_t^T l_s(X_s^{t,x}, u_s)\, dA_s + g(X_T^{t,x}) \right],$$

where $\mathbb{E}_u^{\mathcal{F}_t}$ denotes the conditional expectation under $\mathbb{P}_u$ given $\mathcal{F}_t$. We also introduce the (random) value function

$$v(t, x) = \operatorname*{ess\,inf}_{u(\cdot) \in \mathcal{A}} J_t(x, u(\cdot)), \qquad t \in [0, T],\ x \in K.$$

For every $(t, x) \in [0, T] \times K$ we consider the BSDE

$$Y_s^{t,x} + \int_s^T \int_K Z_r^{t,x}(y)\, q(dr\, dy) = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Z_r^{t,x}(\cdot))\, dA_r, \qquad s \in [t, T]. \tag{4.13}$$

We need the following extended variant of Hypothesis 4.5, where we set $X_{t-}^{t,x} = x$:

Hypothesis 4.9 *For every $(t,x) \in [0,T] \times K$ and every $Z \in \mathcal{L}^{1,0}(p)$ there exists a function $\underline{u}^{Z,t,x} : \Omega \times [t,T] \to U$, measurable with respect to $\mathcal{P}$ and $\mathcal{U}$, such that*

$$f(\omega, s, X^{t,x}_{s-}(\omega), Z_s(\omega,\cdot)) = l_t(X_{s-}(\omega), \underline{u}^{Z,t,x}(\omega,s)) + \int_K Z_s(\omega,y) \Big(r_s(\omega,y,\underline{u}^{Z,t,x}(\omega,s)) - 1 \Big) \phi_s(\omega,dy)$$

for almost all $(\omega,s) \in \Omega \times [t,T]$ with respect to the measure $dA_s(\omega)\mathbb{P}(d\omega)$.

This holds for instance if U is a compact metric space and the functions $r_t(\omega,x,\cdot), l_t(\omega,x,\cdot) : U \to \mathbb{R}$ are continuous for every $\omega \in \Omega$, $t \in [0,T]$, $x \in K$.

In this situation Theorem 3.4 can still be applied to find a unique solution $(Y^{t,x}_s, Z^{t,x}_s)_{s\in[t,T]}$. Let us now extend the process $Z^{t,x}$ setting $Z^{t,x}_s = 0$ for $s \in [0,t)$. The corresponding admissible control $\underline{u}^{Z,t,x}$, whose existence is required in Hypothesis 4.9, will be denoted $u^{*,t,x}$ (we set $u^{*,t,x}(\omega,s)$ equal to an arbitrary constant element of U for $s \in [0,t)$).

Theorem 4.10 *Assume that Hypotheses 4.1 and 4.9 are satisfied and that*

$$\mathbb{E} \exp\big((3 + C_r^4) A_T\big) < \infty. \tag{4.14}$$

Suppose also that there exists β such that

$$\beta > \sup |r-1|^2, \quad \mathbb{E}\exp(\beta A_T) < \infty, \quad \mathbb{E}[|g(X^{t,x}_T)|^2 e^{\beta A_T}] < \infty, \qquad t \in [0,T],\ x \in K, \tag{4.15}$$

(in particular, (4.15) follows from (4.14) with $\beta = 3 + C_r^4$ if g is bounded). For any $(t,x) \in [0,T] \times K$ let $(Y^{t,x}, Z^{t,x})$ denote the solution of the BSDE (4.13) and $u^{*,t,x} = \underline{u}^{Z,t,x}$ the corresponding admissible control.

Then $u^*(\cdot)$ is optimal and $Y^{t,x}_t$ is the optimal cost, i.e. $Y^{t,x}_t = J_t(x, u^*(\cdot)) = v(t,x)$ $\mathbb{P}$-a.s.

The proof of Theorem 4.10 is entirely analogous to the proof of Theorem 4.6, the only difference being that in the BSDE one takes the conditional expectation $\mathbb{E}^{\mathcal{F}_t}_u$ instead of the expectation $\mathbb{E}_u$.

Remark 4.11

1. Let $u \in \mathcal{A}$. Then, under $\mathbb{P}_u$, the compensator of the process N is A^u defined in (4.4). It might therefore be more natural to define as the cost corresponding to $u \in \mathcal{A}$ the functional

$$\mathbb{E}_u \left[\int_0^T l_t(X_t, u_t)\, dA^u_t + g(X_T) \right] = \mathbb{E}_u \left[\int_0^T l_t(X_t, u_t) \int_K r_t(y, u_t)\, \phi_t(dy)\, dA_t + g(X_T) \right],$$

   instead of $J(u(\cdot))$. This cost functional has the same form as $J(u(\cdot))$, with the function l replaced by $l^0_t(x,u) := l_t(x,u) \int_K r_t(y,u)\, \phi_t(dy)$. Since l^0 is $\mathcal{P} \otimes \mathcal{K} \otimes \mathcal{U}$-measurable and bounded, the statements of Theorems 4.6 and 4.10 remain true without any change.

2. Suppose that the cost functional has the form

$$J^1(u(\cdot)) = \mathbb{E}_u \left[\sum_{n\geq 1 : T_n \leq T} c(T_n, X_{T_n}, u_{T_n}) \right],$$

   for some given function $c : \Omega \times [0,T] \times K \times U \to \mathbb{R}$ which is assumed to be bounded and $\mathcal{P} \otimes \mathcal{K} \otimes \mathcal{U}$-measurable. It is well known (see e.g. [6], chapter VII, §1, remark (β)) that we can reduce this control problem to the previous one noting that

$$J^1(u(\cdot)) = \mathbb{E}_u \int_0^T \int_K c(t,y,u_t)\, p(dt\, dy) = \mathbb{E}_u \int_0^T \int_K c(t,y,u_t)\, r_t(y,u_t)\, \phi_t(dy)\, dA_t.$$

Thus, $J^1(u(\cdot))$ has the same form as $J(u(\cdot))$, with $g = 0$ and the function l replaced by $l^1_t(x,u) := \int_K c(t,y,u)\, r_t(y,u)\, \phi_t(dy)$. Since l^1 is $\mathcal{P}\otimes\mathcal{K}\otimes\mathcal{U}$-measurable and bounded, Theorems 4.6 and 4.10 can still be applied.

Similar considerations obviously hold for cost functionals of the form $J(u(\cdot)) + J^1(u(\cdot))$.

5 The stochastic Hamilton-Jacobi-Bellman equation

Throughout this section we still assume that a marked point process is given, satisfying the assumptions of Section 2. In particular we suppose that $T_n \to \infty$ $\mathbb{P}$-a.s. and that (2.1) holds.

We address the same optimal control problem as in the previous section. The associated stochastic Hamilton-Jacobi-Bellman equation (HJB for short) is a backward stochastic differential equation for unknown random fields on $[0,T]\times K$, having the Hamiltonian function defined in (4.8) as a nonlinear term. Before introducing the HJB equation we need a preliminary result which may have an interest in its own and will be used to clarify the connections with the optimal control problem and the BSDEs introduced in the previous section, as well as in the proof of the main result, Theorem 5.4.

5.1 A lemma of Ito type

The Ito formula for processes defined by stochastic integrals with respect to random measures is certainly known, see e.g. [16]: it gives a canonical decomposition of $v(t,X_t)$ for a deterministic functions $v(t,x)$ smooth enough. We need an extension to the case when $v(t,x)$ is stochastic and itself defined by integrals with respect to random measures. The following result is therefore the analogue to the so-called Ito-Kunita formula (also attributed to Bismut and Wentzell, see e.g. [3], [25],[19]).

Lemma 5.1 *Assume that $v, f : \Omega\times[0,T]\times K \to \mathbb{R}$ are $Prog\otimes\mathcal{K}$-measurable, $V : \Omega\times[0,T]\times K\times K \to \mathbb{R}$ is $\mathcal{P}\otimes\mathcal{K}\otimes\mathcal{K}$-measurable, and, $\mathbb{P}$-a.s.*

$$\int_0^T |f(t,x)|\, dA_t + \int_0^T\int_K |V(t,x,y)|\, \phi_t(dy)\, dA_t < \infty, \qquad x\in K. \tag{5.1}$$

Suppose that, $\mathbb{P}$-a.s.

$$v(t,x) - v(0,x) = \int_0^t f(s,x)\, dA_s + \int_0^t\int_K V(s,x,y)\, q(ds\, dy), \qquad t\in[0,T],\ x\in K. \tag{5.2}$$

Then, $\mathbb{P}$-a.s.

$$\begin{aligned} v(t,X_t) - v(0,X_0) \ &= \int_0^t f(s,X_s)\, dA_s + \int_0^t\int_K \Big(v(s-,y) - v(s-,X_{s-}) + V(s,y,y)\Big)\, p(ds\, dy) \\ &\quad - \int_0^t\int_K V(s,X_s,y)\, \phi_s(dy)\, dA_s, \qquad t\in[0,T],\ x\in K. \end{aligned} \tag{5.3}$$

If, in addition,

$$\int_0^T\int_K |v(t,y) + V(t,y,y)|\, \phi_t(dy)\, dA_t < \infty, \qquad \mathbb{P}-a.s.$$

then, $\mathbb{P}$*-a.s.*

$$\begin{aligned} v(t, X_t) - v(0, X_0) \;=\; & \int_0^t f(s, X_s)\, dA_s + \int_0^t \int_K \Big(v(s-, y) - v(s-, X_{s-}) + V(s, y, y) \Big)\, q(ds\, dy) \\ & + \int_0^t \int_K \Big(v(s, y) - v(s, X_s) + V(s, y, y) - V(s, X_s, y) \Big)\, \phi_s(dy)\, dA_s, \end{aligned} \tag{5.4}$$

for every $t \in [0, T]$*,* $x \in K$*.*

Remark 5.2 1. It follows from (5.2) that $\mathbb{P}$-a.s. the trajectories $v(\cdot, x)$ are cadlag for every $x \in K$. Therefore the process $(v(t-, x))$ is well defined and $\mathcal{P} \otimes \mathcal{K}$-measurable.

2. We note that

$$\int_0^T \int_K |V(t, X_t, y)|\, \phi_t(dy)\, dA_t = \sum_{n \geq 1} \int_{T_{n-1} \wedge T}^{T_n \wedge T} \int_K |V(t, \xi_{n-1}, y)|\, \phi_t(dy)\, dA_t < \infty, \quad \mathbb{P} - a.s.$$

This follows from the assumption (5.1), and the fact that the sum is finite $\mathbb{P}$-a.s. due to the assumption that $T_n \to \infty$. Similarly,

$$\int_0^T |f(t, X_t)|\, dA_t + \int_0^T |v(t, X_t)|\, dA_t < \infty, \qquad \mathbb{P} - a.s.$$

so that all the integrals above are well defined: compare the discussion in subsection 2.3.

Proof. Noting that there are N_t jump times T_n in the time interval $[0, t]$ we have

$$v(t, X_t) - v(0, X_0) = \sum_{n=1}^{N_t} \Big(v(T_n-, X_{T_n}) - v(T_{n-1}-, X_{T_{n-1}}) \Big) + v(t, X_t) - v(T_{N_t}-, X_{T_{N_t}}),$$

where we use the convention $v(0-, x) = v(0, x)$. Since $X_t = X_{T_{N_t}}$ we have

$$v(t, X_t) - v(0, X_0) = I + II,$$

where

$$I = \sum_{n=1}^{N_t} \Big(v(T_n-, X_{T_n}) - v(T_n-, X_{T_{n-1}}) \Big),$$

$$II = \sum_{n=1}^{N_t} \Big(v(T_n-, X_{T_{n-1}}) - v(T_{n-1}-, X_{T_{n-1}}) \Big) + v(t, X_{T_{N_t}}) - v(T_{N_t}-, X_{T_{N_t}}).$$

Letting H denote the $\mathcal{P} \otimes \mathcal{K}$-measurable process

$$H_s(y) = v(s-, y) - v(s-, X_{s-}),$$

with the convention $X_{0-} = X_0$, we have

$$I = \sum_{n \geq 1 : T_n \leq t} \Big(v(T_n-, X_{T_n}) - v(T_n-, X_{T_{n-1}}) \Big) = \sum_{n \geq 1 : T_n \leq t} H_{T_n}(X_{T_n}) = \int_0^t \int_K H_s(y)\, p(ds\, dy).$$

For $n = 1, \ldots, N_t$, recalling that $q(dt\, dy) = p(dt\, dy) - \phi_t(dy)\, dA_t$ and the definition of p,

$$\begin{aligned} & v(T_n-, x) - v(T_{n-1}-, x) \\ & \quad = V(T_{n-1}, x, \xi_{n-1}) - \int_{T_{n-1}}^{T_n} \int_K V(s, x, y)\, \phi_s(dy)\, dA_s + \int_{T_{n-1}}^{T_n} f(s, x)\, dA_s \end{aligned}$$

Setting $x = X_{T_{n-1}} = \xi_{n-1}$, noting that $X_s = X_{T_{n-1}}$ for $s \in (T_{n-1}, T_n)$ and recalling that A is assumed to be continuous,

$$\begin{aligned}
&v(T_n-, X_{T_{n-1}}) - v(T_{n-1}-, X_{T_{n-1}}) \\
&\quad = V(T_{n-1}, \xi_{n-1}, \xi_{n-1}) - \int_{T_{n-1}}^{T_n} \int_K V(s, X_s, y)\, \phi_s(dy)\, dA_s + \int_{T_{n-1}}^{T_n} f(s, X_s)\, dA_s.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&v(t, X_{T_{N_t}}) - v(T_{N_t}-, X_{T_{N_t}}) \\
&\quad = V(T_{N_t}, \xi_{N_t}, \xi_{N_t}) - \int_{T_{N_t}}^{t} \int_K V(s, X_s, y)\, \phi_s(dy)\, dA_s + \int_{T_{N_t}}^{t} f(s, X_s)\, dA_s.
\end{aligned}$$

It follows that

$$\begin{aligned}
II &= \sum_{n \geq 1: T_n \leq t} V(T_n, \xi_n, \xi_n) - \int_0^t \int_K V(s, X_s, y)\, \phi_s(dy)\, dA_s + \int_0^t f(s, X_s)\, dA_s \\
&= \int_0^t \int_K V(s, y, y)\, p(ds\, dy) - \int_0^t \int_K V(s, X_s, y)\, \phi_s(dy)\, dA_s + \int_0^t f(s, X_s)\, dA_s,
\end{aligned}$$

and (5.3) is proved. Using again the equality $q(dt\, dy) = p(dt\, dy) - \phi_t(dy)\, dA_t$ and the additional assumption, (5.4) follows as well. □

Remark 5.3 In differential form, under the assumptions of the lemma, if

$$dv(t, x) = f(t, x)\, dA_t + \int_K V(t, x, y)\, q(dt\, dy)$$

then

$$\begin{aligned}
dv(t, X_t) &= f(t, X_t)\, dA_t + \int_K \Big(v(t-, y) - v(t-, X_{t-}) + V(t, y, y) \Big)\, q(dt\, dy) \\
&\quad + \int_K \Big(v(t, y) - v(t, X_t) + V(t, y, y) - V(t, X_t, y) \Big)\, \phi_t(dy)\, dA_t.
\end{aligned}$$

5.2 The equation

In the rest of this section we will suppose that U, l, r, g are given satisfying Hypotheses 4.1 and 4.5 as before. For technical reasons we will also assume that the space K is finite or countable (and $\mathcal{K}$ is the collection of all its subsets). We next present the HJB equation by first introducing the space of processes where we seek its solution.

A pair (v, V) is said to belong to the space $\mathbb{H}_\beta$, where $\beta \in \mathbb{R}$, if

1. $v : \Omega \times [0, T] \times K \to \mathbb{R}$ is $Prog \otimes \mathcal{K}$-measurable, $V : \Omega \times [0, T] \times K \times K \to \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{K} \otimes \mathcal{K}$-measurable;

2. The following is finite:

$$\begin{aligned}
|||(v, V)|||_\beta^2 &:= \sup_{x \in K} \mathbb{E} \int_0^T v(t, x)^2 e^{\beta A_t} dA_t + \mathbb{E} \int_0^T v(t, X_t)^2 e^{\beta A_t} dA_t \\
&\quad + \sup_{x \in K} \mathbb{E} \int_0^T \int_K V(t, x, y)^2 \phi_t(dy)\, e^{\beta A_t} dA_t \\
&\quad + \mathbb{E} \int_0^T \int_K |v(t, y) + V(t, y, y)|^2 \phi_t(dy)\, e^{\beta A_t} dA_t.
\end{aligned}$$

The space $\mathbb{H}_\beta$, endowed with the norm $|||\cdot|||_\beta$, is Banach space, provided we identify pairs of processes whose difference has norm zero.

Let f be the Hamiltonian function defined in (4.8). A pair $(v,V)\in\mathbb{H}_\beta$ is called a solution to the stochastic HJB equation if, $\mathbb{P}$-a.s.,

$$\begin{aligned} v(t,x) &+ \int_t^T \int_K V(s,x,y)\, q(ds\, dy) \\ &= g(x) + \int_t^T \int_K \Big(v(s,y) - v(s,x) + V(s,y,y) - V(s,x,y) \Big)\, \phi_s(dy)\, dA_s \\ &+ \int_t^T f\Big(s,x,v(s,\cdot) - v(s,x) + V(s,\cdot,\cdot)\Big)\, dA_s, \qquad t\in[0,T],\, x\in K. \end{aligned} \tag{5.5}$$

We will also use the differential notation:

$$\left\{ \begin{array}{lcl} -dv(t,x) & = & -\displaystyle\int_K V(t,x,y)\, q(dt\, dy) \\ & & +\displaystyle\int_K \Big(v(t,y) - v(t,x) + V(t,y,y) - V(t,x,y) \Big)\, \phi_t(dy)\, dA_t \\ & & +f\Big(t,x,v(t,\cdot) - v(t,x) + V(t,\cdot,\cdot)\Big)\, dA_t, \\ v(T,x) & = & g(x), \qquad t\in[0,T],\, x\in K. \end{array} \right.$$

The basic result, which we assume for the moment and we will prove later, is the following. Let $\beta_0 > 1$ satisfy

$$\frac{2(2L^2+3)}{\beta_0 - 1} + \frac{8(2L^2+3)}{\beta_0}\left(1 + \frac{1}{\beta_0}\right) < 1.$$

Theorem 5.4 *Let K be finite or countable and let Hypotheses 4.1 and 4.9 be verified. Suppose that there exists β such that*

$$\beta \ge \beta_0, \qquad \sup_{x\in K} \mathbb{E}\left[g(x)^2 e^{\beta A_T} \right] < \infty. \tag{5.6}$$

Then the HJB equation has a unique solution (v,V) in $\mathbb{H}_\beta$.

5.3 Application to control problems and BSDEs

For every $(t,x)\in[0,T]\times K$ we consider again the optimal control problem described just before Theorem 4.10 and the BSDE (4.13) for the unknown processes $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$.

Let $(v,V)\in\mathbb{H}_\beta$ the solution to the HJB equation constructed in Theorem 5.4.

Then we obtain the following result.

Theorem 5.5 *We make the same assumptions as in Theorem 5.4, assuming in addition that β also satifies* (4.15)*. Then for every $(t,x)\in[0,T]\times K$ we have*

$$Y_s^{t,x} = v(s, X_s^{t,x}), \qquad Z_s^{t,x}(y) = v(s-,y) - v(s-, X_{s-}^{t,x}) + V(s,y,y). \tag{5.7}$$

In particular, $v(t,x) = Y_t^{t,x}$ $\mathbb{P}$-a.s.

If (4.14) *also holds then $v(t,x)$ coincides with the value function of the optimal control problem i.e. $v(t,x) = \operatorname{ess\,inf}_{u(\cdot)\in\mathcal{A}} J_t(x,u(\cdot))$ $\mathbb{P}$-a.s.*

Equalities (5.7) should be understood up to sets of measure zero in $\Omega \times [t, T]$, the measure being $dA_s(\omega)\mathbb{P}(d\omega)$ for the first equality and $\phi_s(\omega, dy)dA_s(\omega)\mathbb{P}(d\omega)$ for the second equality.

Proof. We use a straightforward extension of the Ito lemma 5.1 to compute the stochastic differential $dv(s, X_s^{t,x})$ on the interval $[t, T]$ instead of $[0, T]$. Using the Lipschitz character of f it is not difficult to check that all the assumptions of the lemma are verified. For instance, we check that for every $x \in K$

$$\mathbb{E}\int_0^T \int_K |V(t,x,y)| \;\phi_t(dy)\, dA_t \le \left(\mathbb{E}\int_0^T \int_K |V(t,x,y)|^2 \;\phi_t(dy)\, e^{\beta A_t} dA_t\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T e^{-\beta A_t} dA_t\right)^{\frac{1}{2}}$$

is finite, since $(v, V) \in \mathbb{H}_\beta$ and $\int_0^T e^{-\beta A_t} dA_t = \beta^{-1}(1 - e^{-\beta A_T}) \le \beta^{-1}$, so that V satisfies the required condition (5.1). The other verifications are similar and are therefore omitted.

The Ito lemma then yields

$$\begin{aligned} &v(s, X_s^{t,x}) + \int_s^T \int_K \Big(v(r-,y) - v(r-, X_{r-}^{t,x}) + V(r,y,y)\, q(dr\, dy) \\ &= g(X_T^{t,x}) + \int_s^T f_r(X_r^{t,x}, v(r-,\cdot) - v(r-, X_{r-}^{t,x}) + V(r,\cdot,\cdot))\, dA_r, \qquad s \in [t,T]. \end{aligned}$$

Comparing with equation (4.13) and setting

$$\tilde{Y}_s^{t,x} = v(s, X_s^{t,x}), \qquad \tilde{Z}_s^{t,x} = v(s-,y) - v(s-, X_{s-}^{t,x}) + V(s,y,y),$$

we conclude that the pairs $(Y_s^{t,x}, Z_s^{t,x})$ and $(\tilde{Y}_s^{t,x}, \tilde{Z}_s^{t,x})$ are solutions to the same BSDE, and the latter also belongs to $\mathbb{K}^\beta$ as it follows easily from the fact that (v, V) belongs to $\mathbb{H}_\beta$. By uniqueness for the solution to the BSDE, (5.7) holds.

All the other statements follow from Theorem 4.10. □

5.4 Proof of Theorem 5.4

It is convenient to first state the following simple preliminary result.

Lemma 5.6 *Suppose*

$$-dv(t,x) = -\int_K V(t,x,y)\, q(dt\, dy) + \int_K U(t,x,y)\, \phi_t(dy)\, dA_t + u(t,x)\, dA_t, \quad v(T,x) = g(x).$$

Then, setting $c_\beta = \frac{2}{\beta - 1}$ *for* $\beta > 1$, *we have, for every* $x \in K$,

$$\begin{aligned} &\mathbb{E}\int_0^T v(s,x)^2 e^{\beta A_s} dA_s + \mathbb{E}\int_0^T \int_K V(s,x,y)^2 \phi_s(dy)\, e^{\beta A_s} dA_s \\ &\le \mathbb{E}\left[g(x)^2 e^{\beta A_T}\right] + c_\beta \mathbb{E}\int_0^T u(s,x)^2\, e^{\beta A_s} dA_s + c_\beta \mathbb{E}\int_0^T \int_K U(t,x,y)^2\, \phi_s(dy)\, e^{\beta A_s} dA_s. \end{aligned}$$

Proof. Using the identity (3.7) of Lemma 3.3 we have

$$\begin{aligned} &\mathbb{E}\left[v(t,x)^2 e^{\beta A_t}\right] + \beta \mathbb{E}\int_t^T v(s,x)^2 e^{\beta A_s} dA_s + \mathbb{E}\int_t^T \int_K V(s,x,y)^2 \phi_s(dy)\, e^{\beta A_s} dA_s \\ &\quad = \mathbb{E}\left[g(x)^2 e^{\beta A_T}\right] + 2\mathbb{E}\int_t^T v(s,x) \left[\int_K U(t,x,y)\, \phi_s(dy) + u(s,x)\right] e^{\beta A_s} dA_s. \end{aligned}$$

Setting $t = 0$ and using the elementary inequality

$$2v(s,x)\left[\int_K U(t,x,y)\,\phi_s(dy) + u(s,x)\right] \le (\beta-1)v(s,x)^2 + c_\beta\left[\int_K U(t,x,y)^2\,\phi_s(dy) + u(s,x)^2\right]$$

the conclusion follows immediately. □

Proof of Theorem 5.4. We define a map $\Gamma : \mathbb{H}_\beta \to \mathbb{H}_\beta$ setting $(v,V) = \Gamma(u,U)$, for $(u,U) \in \mathbb{H}_\beta$, if (v,V) is the solution of

$$\begin{cases} -dv(t,x) &= -\displaystyle\int_K V(t,x,y)\,q(dt\,dy) \\ &\quad + \displaystyle\int_K \Big(u(t,y)-u(t,x)+U(t,y,y)-V(t,x,y)\Big)\,\phi_t(dy)\,dA_t \\ &\quad + f\Big(t,x,u(t,\cdot)-u(t,x)+U(t,\cdot,\cdot)\Big)\,dA_t, \\ v(T,x) &= g(x), \qquad t\in[0,T],\ x\in K. \end{cases}$$

Note the two occurrences of V in the right-hand side. For fixed $x \in K$, the existence of processes $v(\cdot,x), V(\cdot,x,\cdot)$ solution to this equation follows from an application of Theorem 3.4. Since K is assumed to be at most countable, the corresponding integral equation holds simultaneously for every $t \in [0,T]$ and $x \in K$, with the exception of a $\mathbb{P}$-null set. The rest of the proof consists in showing that $(v,V) \in \mathbb{H}_\beta$ and that Γ is a contraction for sufficiently large β. We limit ourselves to proving the contraction property, since the fact that $(v,V) \in \mathbb{H}_\beta$ can be verified by similar and simpler arguments.

Let $(u^i,U^i) \in \mathbb{H}_\beta$ for $i = 1,2$ and let $(v^i,V^i) = \Gamma(u^i,U^i)$. Define $\bar v = v^2 - v^1$, $\bar V = V^2 - V^1$, $\bar u = u^2 - u^1$, $\bar U = U^2 - U^1$,

$$\bar f(t,x) = f\Big(t,x,u^2(t,\cdot)-u^2(t,x)+U^2(t,\cdot,\cdot)\Big) - f\Big(t,x,u^1(t,\cdot)-u^1(t,x)+U^1(t,\cdot,\cdot)\Big).$$

Then

$$\begin{cases} -d\bar v(t,x) &= -\displaystyle\int_K \bar V(t,x,y)\,q(dt\,dy) \\ &\quad + \displaystyle\int_K \Big(\bar u(t,y)-\bar u(t,x)+\bar U(t,y,y)-\bar V(t,x,y)\Big)\,\phi_t(dy)\,dA_t + \bar f(t,x)\,dA_t, \\ v(T,x) &= 0, \qquad t\in[0,T],\ x\in K. \end{cases} \tag{5.8}$$

From Lemma 5.6 it follows that, for every $x \in K$, $\beta > 1$

$$\mathbb{E}\int_0^T \bar v(s,x)^2 e^{\beta A_s}\,dA_s + \mathbb{E}\int_0^T\int_K \bar V(s,x,y)^2\phi_s(dy)\,e^{\beta A_s}\,dA_s \le \frac{2}{\beta-1}\mathbb{E}\int_0^T \bar f(s,x)^2\,e^{\beta A_s}\,dA_s$$
$$+\frac{2}{\beta-1}\mathbb{E}\int_0^T\int_K \left[\bar u(s,y)-\bar u(s,x)+\bar U(s,y,y)-\bar V(s,x,y)\right]^2\,\phi_s(dy)\,e^{\beta A_s}\,dA_s.$$

By the Lipschitz condition on f we have

$$\bar f(s,x)^2 \le L^2\int_K \left[\bar u(s,y)-\bar u(s,x)+\bar U(s,y,y)\right]^2\,\phi_s(dy), \tag{5.9}$$

and it follows that, for every $x \in K$, $\beta > 1$

$$\mathbb{E}\int_0^T \bar v(s,x)^2 e^{\beta A_s}\,dA_s + \mathbb{E}\int_0^T\int_K \bar V(s,x,y)^2\phi_s(dy)\,e^{\beta A_s}\,dA_s$$
$$\le \frac{2(2L^2+3)}{\beta-1}\left(\mathbb{E}\int_0^T \bar u(s,x)^2\,e^{\beta A_s}\,dA_s + \mathbb{E}\int_0^T\int_K \left[\bar u(s,y)+\bar U(s,y,y)\right]^2\,\phi_s(dy)\,e^{\beta A_s}\,dA_s\right)$$
$$+\frac{6}{\beta-1}\mathbb{E}\int_0^T\int_K \bar V(s,x,y)^2\,\phi_s(dy)\,e^{\beta A_s}\,dA_s.$$

Setting $c_\beta^{(1)} := \frac{2(2L^2+3)}{\beta-1}$ it follows that

$$\sup_{x\in K} \mathbb{E}\int_0^T \bar{v}(s,x)^2 e^{\beta A_s} dA_s + \sup_{x\in K} \mathbb{E}\int_0^T \int_K \bar{V}(s,x,y)^2 \phi_s(dy)\, e^{\beta A_s} dA_s \le c_\beta^1\, |||(\bar{u},\bar{U})|||_\beta^2. \quad (5.10)$$

We set now

$$\bar{Y}_s = \bar{v}(s, X_s), \quad \bar{Z}_s(y) = \bar{v}(s-, y) - \bar{v}(s-, X_{s-}) + \bar{V}(s,y,y).$$

Recalling (5.8) and applying the Ito formula of Lemma 5.1 we obtain

$$\begin{cases} d\bar{Y}_t &= \displaystyle\int_K \bar{Z}_t(y)\, q(dt\, dy) - \bar{f}(t, X_t)\, dA_t \\ & \quad + \displaystyle\int_K \Big(\bar{Z}_t(y) - \bar{u}(t,y) + \bar{u}(t,X_t) - \bar{U}(t,y,y)\Big)\, \phi_t(dy)\, dA_t, \end{cases}$$

and $\bar{Y}_T = 0$. Note that the term $\bar{V}(t, X_t, y)$ has disappeared. Using the estimate (3.8) in Lemma 3.3 on the BSDE we have

$$\mathbb{E}\int_0^T \bar{Y}_s^2 e^{\beta A_s} dA_s + \mathbb{E}\int_0^T \int_K \bar{Z}_s(y)^2 \phi_s(dy)\, e^{\beta A_s} dA_s \le \frac{8}{\beta}\left(1+\frac{1}{\beta}\right) \mathbb{E}\int_0^T \bar{f}(s,X_s)^2\, e^{\beta A_s} dA_s$$
$$+\frac{8}{\beta}\left(1+\frac{1}{\beta}\right) \mathbb{E}\int_0^T \int_K \left[\bar{Z}_s(y) - \bar{u}(s,y) + \bar{u}(s,X_s) - \bar{U}(s,y,y)\right]^2\, \phi_s(dy)\, e^{\beta A_s} dA_s.$$

Using again the inequality (5.9) we obtain

$$\begin{aligned} \mathbb{E}\int_0^T \bar{Y}_s^2 e^{\beta A_s} dA_s &+ \mathbb{E}\int_0^T \int_K \bar{Z}_s(y)^2 \phi_s(dy)\, e^{\beta A_s} dA_s \\ &\le \frac{8(2L^2+3)}{\beta}\left(1+\frac{1}{\beta}\right) \mathbb{E}\int_0^T \bar{u}(s,X_s)^2\, e^{\beta A_s} dA_s \\ &+ \frac{8(2L^2+3)}{\beta}\left(1+\frac{1}{\beta}\right) \mathbb{E}\int_0^T \int_K \left[\bar{u}(s,y) + \bar{U}(s,y,y)\right]^2\, \phi_s(dy)\, e^{\beta A_s} dA_s \\ &+ \frac{24}{\beta}\left(1+\frac{1}{\beta}\right) \mathbb{E}\int_0^T \int_K \bar{Z}_s(y)^2\, \phi_s(dy)\, e^{\beta A_s} dA_s. \end{aligned}$$

Setting $c_\beta^{(2)} := \frac{8(2L^2+3)}{\beta}\left(1+\frac{1}{\beta}\right)$ it follows that

$$\mathbb{E}\int_0^T \bar{Y}_s^2 e^{\beta A_s} dA_s + \mathbb{E}\int_0^T \int_K \bar{Z}_s(y)^2 \phi_s(dy)\, e^{\beta A_s} dA_s \le c_\beta^2\, |||(\bar{u},\bar{U})|||_\beta^2. \quad (5.11)$$

Recalling the definition of $\bar{Y}, \bar{Z}$ and using the fact that A is assumed to be continuous we have

$$\mathbb{E}\int_0^T \int_K \left[\bar{v}(s,y) + \bar{V}(s,y,y)\right]^2\, \phi_s(dy)\, e^{\beta A_s} dA_s = \mathbb{E}\int_0^T \int_K \left[\bar{Z}_s(y) + \bar{Y}_s\right]^2 \phi_s(dy)\, e^{\beta A_s} dA_s$$
$$\le 2\mathbb{E}\int_0^T \bar{Y}_s^2 e^{\beta A_s} dA_s + 2\mathbb{E}\int_0^T \int_K \bar{Z}_s(y)^2 \phi_s(dy)\, e^{\beta A_s} dA_s \le c_\beta^2\, |||(\bar{u},\bar{U})|||_\beta^2, \quad (5.12)$$

where the last inequality is due to (5.11). Recalling that $\bar{Y}_s = \bar{v}(s, X_s)$, it follows from (5.10), (5.11), (5.12) that $|||(\bar{v},\bar{V})|||_\beta^2 \le c_\beta\, |||(\bar{u},\bar{U})|||_\beta^2$ where $c_\beta = c_\beta^{(1)} + c_\beta^{(2)}$ is < 1 by the assumptions. This proves the required contraction property and finishes the proof. □

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